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NUMBER-THEORETICAL ALMOST-PERIODICITIES.*

By Aurel Wintner.

The present paper deals with an harmonic analysis of various types of "hidden periodicities" in the theory of whole numbers. The "periodicities" in question, the simplest manifestations of which have so much fascinated the Greeks, are not of course periodicities proper, and so it is remarkable that the corresponding arithmetical recurrences possess at all an harmonic analysis in the technical sense of the term. In fact, the "recurrences" prove to correspond to an almost-periodic behavior in the sense (B) of Besicovich. It is revealing for the whole situation that, for instance, Bohr's class of uniformly almost-periodic sequences turns out to be all too restrictive to represent the effect of arithmetical fluctuations, except in trivial cases (cases which cannot even occur in connection with the "Eulerian" and "Dirichletian" algorithms, treated in **5** and **9** respectively).

The emphasis in the results obtained is always on the circumstance that the resulting expansions into *trigonometric series,* instead of claiming formal identities, represent *Fourier expansions* for the arithmetical sequences to which they belong. Otherwise the results would have nothing to do with an *harmonic analysis* of the "hidden periodicities" in arithmetics.

1. By a function h will be meant a function $h = h(n)$ or $h = h_m$ of a positive integer, that is, a sequence $h(1), h(2), \cdots$ or $h_1, h_2, \cdots$. The symbol $M(h)$ will denote the mean-value

$$M(h) = \lim_{n\to\infty} (1/n) \sum_{m=1}^{n} h(m), \tag{1}$$

provided that (1) exists as a *finite* limit.

Consider a linear transformation

$$f(n) = \sum_{m=1}^{\infty} e_m(n) g(m) \tag{2}$$

of an arbitrary function g into a function f. The latter is defined for an arbitrary g if and only if the matrix of (2) is "zeilenfinit" (Toeplitz), that is to say such that there exists a function $N = N_m$ having the property that

$$e_m(n) = 0 \quad \text{whenever} \quad n > N_m, \qquad (m = 1, 2, \cdots). \tag{3}$$

* Received February 14, 1944.

For instance, (3) is satisfied by $N_m = m$ if the matrix of (2) is "triangular".

Suppose that the matrix of (2) satisfies (3) and is such that its elements contained in any fixed column form an almost-periodic function (B) of the index of the row, that is, that the functions

$$(4) \qquad e_1(n), e_2(n), \cdots, e_m(n), \cdots \text{ are almost-periodic } (B)$$

(the variable is n, while m is arbitrarily fixed). Then the absolute value of each of the functions (4) of n is almost-periodic (B). In particular, the mean-value $M(|e_m|)$, defined by the case $h = e_m$ of (1), exists for $m = 1, 2, \cdots$.

If the matrix of the linear transformation (2) satisfies (3) and (4), it is easy to verify that any function g satisfying

$$(5) \qquad \sum_{m=1}^{\infty} M(|e_m|)|g(m)| < \infty$$

is transformed by (2) into a function $f(n)$ which is almost-periodic (B) and such that every Fourier constant of $f(n)$ is the limit, as $k \to \infty$, of the corresponding Fourier constant of $f_k(n)$, where

$$(6) \qquad f_k(n) = \sum_{m=1}^{k} e_m(n) g(m);$$

in fact, f_k will tend, as $k \to \infty$, to f in the mean of the (B)-space (cf. [11], pp. 24-25).

Incidentally, it is clear from the proof of these facts that the full force of the assumption (3), an assumption which is chosen independent of g, is not needed.

2. Suppose that the Fourier series (B) of each of the functions (4) is periodic and that the m-th of these Fourier series (B) has the period m, where $m = 1, 2, \cdots$. This means that there exist m constants $\alpha_m(1), \cdots, \alpha_m(m)$ satisfying

$$(7) \qquad e_m(n) \sim \sum_{j=1}^{m} \alpha_m(j) \epsilon(nj/m),$$

where m is arbitrarily fixed and

$$(8) \qquad \epsilon(x) = \exp(2\pi i x).$$

On the other hand, none of the functions (4) of n is required to be periodic. In other words, (7) is compatible with

$$(9) \qquad e_m(n) \neq e^m(n),$$

if $e^m(n)$ denotes the sum on the right of (7). All that can be said is that, besides (7),

(10) $$e^m(n) \sim \sum_{j=1}^{m} \alpha_m(j)\epsilon(nj/m),$$

since

(11) $$e^m(n) = \sum_{j=1}^{m} \alpha_m(j)\epsilon(nj/m)$$

is a finite, and therefore a uniformly convergent, trigonometric sum in n. Correspondingly, the Fourier constants $\alpha_m(j)$ can be calculated not only by the Fourier inversion

(12) $$m\alpha_m(j) = \sum_{n=1}^{m} e_m(n)\epsilon(-jn/m)$$

of (7) but also by the Fourier inversion

(13) $$m\alpha_m(j) = \sum_{n=1}^{m} e^m(n)\epsilon(-jn/m)$$

of (11).

For the purposes of **5** below, it will be fundamental that the possibility (9) is not excluded by the assumptions of the following theorem:

(I) *If the matrix of a linear transformation* (2) *satisfies* (3) *and the particular case* (7) *of* (4), *and if* g *is any function satisfying* (5), *then the function* f *into which* (2) *transforms* g *is almost-periodic* (B) *and has a limit-periodic Fourier series* (B),

(14) $$f(n) \sim \sum_{r=1}^{\infty} \sum_{1 \leqq s \leqq r}^{(s,r)=1} a_r{}^s\, \epsilon(ns/r),$$

in which the Fourier constants have the values represented by the series

(15) $$a_r{}^s = \sum_{r|k} g(k)\alpha_k(ks/r)$$

(*each of which is absolutely convergent*); *so that, in particular,*

(16) $$M(f) = a_1{}^1 = \sum_{k=1}^{\infty} g(k)\alpha_k(k).$$

It is clear from (7) and from the definition of a Fourier constant that $|\alpha_m(j)| \leqq M(|e_m|)$ for every j. On the other hand, the summation index k in (5) runs through all multiples of r, that is, the series (15) can be written in the form

$$\sum_{n=1}^{\infty} \alpha_{nr}(ns)g(nr).$$

Hence, if the preceding inequality is applied to $m = nr$ and $j = nr$, the series (15) is seen to be majorized by the series

$$\sum_{n=1}^{\infty} M(|e_{nr}|)|g(nr)|.$$

But the latter series is a subseries of the series (5), since r is any fixed positive integer. This proves the parenthetical assertion following (15).

Of the remaining statements of (I), only the explicit representation (14), (15) of the Fourier analysis (B) of f remains to be proved. In fact, the almost-periodicity (B) of f follows from **1**, even if the restriction (10) of (4) is omitted. Correspondingly, the proof of (I) will be complete if it is shown that, in virtue of the restriction (10), the function (6) has a Fourier series (B) of the form

$$(17) \qquad f_k(n) \sim \sum_{r=1}^{\infty} \sum_{1 \leq s \leq r}^{(s,r)=1} a_r{}^s(k) \epsilon(ns/r)$$

(for every fixed k) and that, as $k \to \infty$, each of the Fourier constants of (17) tends to the limit

$$(18) \qquad \lim_{k \to \infty} a_r{}^s(k) = \sum_{n=1}^{\infty} \alpha_{nr}(ns) g(nr),$$

that is, to the value (15). In fact, the remark following (6) implies that (17) and (18) are sufficient for (14) and (15).

Substitution of (10) into (6) gives

$$(19) \qquad f_k(n) \sim \sum_{m=1}^{k} \sum_{j=1}^{m} \alpha_m(j) \epsilon(nj/m).$$

This can be written in the form (17), if all those pairs j, m are collected in (19) for which the quotient j/m has a fixed value s/r, where s and r are relatively prime. In other words, (17) is satisfied by

$$(20) \qquad a_r{}^s(k) = \sum_{j/m = s/r}^{m \leq k} \alpha_m(j).$$

But every pair of summation indices j, m occurring in (20) can be represented in the form $j = ns$, $m = nr$, where n is a *unique* positive integer. It is also clear that this n will attain *all* values $1, 2, \cdots$ as $k \to \infty$ in (20). Hence, (18) follows from (20) and from the fact that, as shown above, the series on the right of (18) is *absolutely* convergent.

This completes the proof of (I).

3. Suppose that the matrix of (2) satisfies the assumptions, (2), (4) and (7), of (I).

Of particular arithmetical interest proves to be the case of those among the matrices which have the property that, for any function g satisfying (5), the $\phi(r)$ coefficients (15) of each of the interior sums in (14) are identical except for *fixed phase factors.* By this is meant that there should exist for every s and every r, where

(21) $\qquad (s, r) = 1$ and $1 \leqq s \leqq r, \qquad (r = 1, 2, \cdots),$

a constant $\alpha_r{}^s$ which depends only on the matrix of (2) and has the property that, on the whole s-range (21) belonging to any fixed r, the Fourier constants (15) are expressible in the form

(22) $$a_r{}^s = \epsilon(-\alpha_r{}^s) b_r,$$

where the coefficients $b_1, b_2, \cdots$ are independent of s, and g in (15) is any function satisfying (5). In other words, the Fourier series (14) should have the particular structure

(23) $$f(n) \sim \sum_{r=1}^{\infty} b_r \beta_r(n),$$

where the trigonometric sums

(24) $$\beta_r(n) = \sum_{1 \leqq s \leqq r}^{(s,r)=1} \epsilon(-\alpha_r{}^s + ns/r)$$

depend only on the matrix of (2).

According to (22) and (15), the matrix of (2) will have this property if and only if there exists for every function $g = g(m)$ satisfying (5) a function $b = b_m$ satisfying

(25) $$b_r = \epsilon(\alpha_r{}^s) \sum_{r|k} g(k) \alpha_k(ks/r),$$

where, for any fixed r, the integer s is restricted only by (21). Clearly, (25) can be written in the form.

(26) $$0 = \sum_{n=1}^{\infty} g(rn) \{\epsilon(\alpha_r{}^s) \alpha_{nr}(ns) - \epsilon(\alpha_r{}^t) \alpha_{nr}(nt)\},$$

where t denotes any fixed s satisfying (21) with reference to a given r. But (26) is required for every g satisfying (5) and, therefore, for every $g = g(m)$ tending sufficiently fast to 0 as $m \to \infty$. Since the expression $\{ \quad \}$ multiplying $g(rn)$ in (26) depends only on the matrix of (2), it follows that every $\{ \quad \}$ in (26) must vanish. But t is a fixed s satisfying (21). Hence, the vanishing of every $\{ \quad \}$ in (26) means that, if r, n is any pair of positive integers, the product $\epsilon(\alpha_r{}^s) \alpha_{nr}(ns)$ has a value independent of the choice of the $\phi(r)$ integers s satisfying (21). If r and nr are denoted by d and m, respectively, then this necessary and sufficient condition for the phase rule (22) appears in the following form:

For every positive integer, m, and for every divisor, d, of m, there exists a value, $\lambda = \lambda_m(d)$, such that

(27) $$\epsilon(\alpha_d{}^s) \alpha_m(sm/d) = \lambda_m(d)$$

holds for all $\phi(d)$ integers satisfying

$$(s,d)=1 \quad \text{and} \quad 1 \leqq s \leqq d, \quad \text{where} \quad d \mid m. \tag{28}$$

This criterion can readily be applied to an explicit determination of the linear transformations (2) in question. In fact, the number of pairs of integers s, d, satisfying (28) for a fixed m is

$$\sum_{d \mid m} \sum_{1 \leqq s \leqq d}^{(s,d)=1} 1 = \sum_{d \mid m} \phi(d) = m.$$

Correspondingly, the Fourier expansion (7) of m terms can be written in the form

$$e_m(n) \sim \sum_{d \mid m} \sum_{1 \leqq s \leqq d}^{(s,d)=1} \alpha_m(sm/d)\epsilon(ns/d), \tag{29}$$

whether (27) be satisfied or not. Consequently, the Fourier constants $\alpha_m(j)$ of the columns (4) of the matrix of (2) satisfy (27) if and only if the Fourier expansions (7) can be represented in the form

$$e_m(n) \sim \sum_{d \mid m} \lambda_m(d) \sum_{1 \leqq s \leqq d}^{(s,d)=1} \epsilon(-\alpha_d{}^s + ns/d),$$

which means, by (24), that

$$e_m(n) \sim \sum_{d \mid m} \lambda_m(d)\beta_d(n). \tag{30}$$

Since (27) reduces (25) to

$$b_r = \sum_{r \mid k} \lambda_k(r) g(k), \tag{31}$$

the result of these considerations may be summarized as follows:

(II) *Let the matrix of a linear transformation* (2) *satisfy the assumptions,* (3), (4) *and* (7), *of* (I). *Then the Fourier series* (14) *will have the particular form* (23), (24) *belonging to a given set of* (*not necessarily real*) *absolute constants* $\alpha_r{}^s$ *if and only if there exist absolute constants* $\lambda_m(n)$ *by means of which the Fourier series of the columns* (4) *of the matrix are expressible in the form* (30); *in which case the coefficients of the Fourier series* (23) *of the almost-periodic* (B) *functions* $f(n)$ *are given by the absolutely convergent series* (31), *where* $g(n)$ *denotes the function, subject to* (5), *which defines* $f(n)$ *by* (2).

4. Perhaps the simplest particular case results if (2) is the linear transformation corresponding to the sieve of Eratosthenes, namely,

$$f(n) = \sum_{d \mid n} f'(d), \tag{32}$$

where

(33) $$g(n) = f'(n).$$

The functional notation (33) in the case (32) of (2) is justified by the fact that the function (33) is uniquely determined by (32), if $f(n)$ is any given function. In fact, if (32) is written in the form (2), then (3) is satisfied by $N_m = m$, and the diagonal element $e_m(m)$ is 1 for every m. Thus, the linear transformation (32) of f' into f has a unique inverse, which proves the assertion.

It is also seen that, if (32) is identified with (2) in virtue of (33), then the m-th of the functions (4) can be described as follows: For every m, the function $e_m(n)$ of n has the period m and is 0 or 1 according as $1 \leqq n < m$ or $n = m$ (in fact, this description of the matrix of (32) is equivalent to the sieve process of Eratosthenes). But (8) shows that this description of the functions (4) is equivalent to the explicit formula

(34) $$e_m(n) = (1/m) \sum_{j=1}^{n} \epsilon(nj/m).$$

According to (7) and (11), this means that

(35) $$\alpha_m(j) = 1/m$$

and

(36) $$e_m(n) = e^m(n).$$

Finally, it is seen from (1), (7) and from the case $j = m$ of (35), that $M(e_m) = 1/m$.

Thus it is clear that, if (2) is given by (32) and (33), then the assertion of (I) can be formulated as follows:

(III) *There exists for every function f a unique function f' by means of which f is representable in the form* (32). *Suppose that the function f' belonging to f satisfies the condition*

(37) $$\sum_{k=1}^{\infty} | f'(k)|/k < \infty.$$

Then the function f is almost-periodic (B) *and has the Fourier series* (B)

(38) $$f(n) \sim \sum_{r=1}^{\infty} a_r c_r(n),$$

where $c_r(n)$ denotes the Ramanujan sum

(39) $$c_r(n) = \sum_{1 \leqq s \leqq r}^{(s,r)=1} \epsilon(ns/r)$$

and the Fourier constants a_r are given by the absolutely convergent series

$$a_r = \sum_{r|k} f'(k)/k; \tag{40}$$

so that, in particular,

$$M(f) = a_1 = \sum_{k=1}^{\infty} f'(k)/k. \tag{41}$$

This particular case of (I) was found in [11], pp. 24-25 and p. 30. Considerations with regard to an "arbitrary" function f which sound like (III) but actually either deal with trigonometric identities, rather than with Fourier series, or postulate, rather than prove, the existence of a Fourier series (38) of some sort are, of course, of a much older date; cf., e. g., the considerations of Libri [8] or Carmichael [2]. Incidentally, it is sufficient to compare (14) in (I) or (23) in (II) with the uniqueness theorem of a Fourier series (B), in order to see how meaningless a postulation of the *specific* expansion (38) in (III) would be. In this connection, cf. a question raised by Hardy [6].

5. Let now the "Eratosthenian" process (32) be replaced by the "Eulerian" process

$$f(n) = \sum_{1 \leq l \leq n}^{(l,n)=1} f^*(l) \tag{42}$$

("Eulerian," since, in virtue of (42),

$$f(n) = \phi(n) \quad \text{when} \quad f^*(n) = 1; \tag{43}$$

incidentally, from (39),

$$f(n) = c_r(n) \quad \text{when} \quad f^*(n) = \epsilon(nr), \tag{44}$$

where r is any fixed positive integer).

It is clear from (42) that $f(1) = f^*(1)$ and $f(2) = f^*(1)$; hence

$$f(1) = f(2). \tag{45}$$

Conversely, if $f = f(n)$ is any function satisfying the initial restriction (45), then there exists a unique function $f^* = f^*(n)$ by means of which f is representable in the form (42). In other words, the functional notation

$$g = f^* \tag{46}$$

is justified if (42) is thought of as written in the form (2), provided that the trivial necessary condition (45) is satisfied. In fact, if $n \geqq 2$, then (l, n) is 1 for $l = n - 1$ but is not 1 for $l = n$. It follows, therefore, from (42)

that, if $n \geqq 2$, then $f(n) = F_{n-2} + f^*(n-1)$, where F_{n-2} denotes the sum of certain of the $n-2$ values $f^*(m)$ belonging to $m = 1, \cdots, n-2$ (with the understanding that the number of these values can be 0, in which case $F_{n-2} = 0$). Consequently, if (45) is satisfied, then, since $f^*(n-1) = f(n) - F_{n-2}$ for $n \geqq 2$ and $f^*(1) = f(1)$, $f^*(2) = f(1)$, the values of f^* can be determined, recursively, from those of f for every n. This proves the assertion.

Let (42) be thought of as written in the form (2), where the function (46) is arbitrary. Then, since F_{n-2} does not contain $f(m)$ when $m > n-1 (> 0)$, condition (3) is satisfied by $N_m = m-1$ if $n \geqq 2$, while $N_1 = 1$. Actually, the sieve process of Eratosthenes, referred to in connection with (32), can readily be adapted to (42) and leads, when (42) is written in the form (2), to the following description of the structure of the column functions (4): For every positive integer m (including $m = 1$),

(47) $e_m(n) = 0$ if $1 \leqq n < m$ and $e_m(n) = e^m(n)$ if $m \leqq n < \infty$,

where $e^m(n)$ denotes 1 or 0 according as n and m are or are not relatively prime (so that, in particular,

(48) $$e_1(n) = e^1(n) = 1$$
for every n).

Clearly, the function $e^m(n)$ just defined is a periodic function of n and has m as a period. Although the primitive period of $e^m(n)$ is in general (namely, unless m is square-free) less than m, it follows that $e^m(n)$ can, for every m, be represented in the form (11). The coefficients $\alpha_m(j)$ of (11) can be calculated from (13). This gives

$$m\alpha_m(j) = \sum_{1 \leqq s \leqq m}^{(s,m)=1} \epsilon(-js/m).$$

But the sum on the right of this relation remains unaltered if $-j$ is replaced by j. Hence, from (39),

(50) $$\alpha_m(j) = c_m(j)/m.$$

It follows, therefore, from (11) that the function $e^m(n)$ occurring in the representation (47) of the matrix of the linear transformation (42) of f^* into f can be given explicitly as follows:

(51) $$e^m(n) = \sum_{j=1}^{m} c_m(j)\epsilon(nj/m)/m.$$

For the column functions (4) themselves, (51) implies the Fourier series (B)

(52) $$e_m(n) \sim \sum_{j=1}^{m} c_m(j)\epsilon(nj/m)/m,$$

by (11), (10) and (7); cf. (9) and (47). In particular,

$$M(|e_m|) = M(e_m) = \phi(m)/m. \tag{53}$$

In fact, the constant term of the Fourier series (52) belongs to $j = m$ and has, therefore, the value $c_m(m)/m$. This means, by (1), that $M(e_m) = c_m(m)/m$. Hence, (53) follows by placing $r = m$ and $n = m$ in (39) and using the fact that $e_m \geqq 0$.

If all of this is combined with (I), there results the following theorem:

(IV) *For an arbitrary function $f = f(n)$ subject to (45), there exists a unique function $f^* = f^*(n)$ by means of which f is representable in the form (42). Suppose that the function f^* belonging to f satisfies the condition*

$$\sum_{k=1}^{\infty} |f^*(k)|\, \phi(k)/k < \infty, \tag{54}$$

where ϕ denotes Euler's function. Then f is almost-periodic (B) and has the Fourier series (B)

$$f(n) \sim \sum_{r=1}^{\infty} a_r c_r(n), \tag{55}$$

where the $c_r(n)$ are the Ramanujan sums (39) and, if μ denotes Möbius' function, the Fourier constants a_r are given by the absolutely convergent series

$$a_r = \frac{\mu(r)}{\phi(r)} \sum_{r|k} \frac{\phi(k)}{k} f^*(k); \tag{56}$$

so that, in particular,

$$M(f) = a_1 = \sum_{k=1}^{\infty} f^*(k)\phi(k)/k \tag{57}$$

and

$$a_r = 0 \text{ unless } r \text{ is square-free.} \tag{58}$$

In fact, if (2) is identified with (42), it is seen from (46) and (53) that the assumption (5) of (I) becomes the assumption (54) of (IV). On the other hand, (46) and (50) show that the assertion (15) of (I) for (14) can now be written in the form

$$a_r{}^s = \sum_{k=1}^{\infty} f^*(rk) c_{rk}(ks)/(rk), \tag{59}$$

where $(s, r) = 1$, by (21) or (14). But it is easily verified from (39) and from the definitions of Euler's ϕ and of Möbius' μ that

$$c_{rk}(ks) = \phi(kr)\mu(r)/\phi(r), \quad \text{where} \quad (s, r) = 1, \tag{60}$$

holds for $k = 1, 2, \cdots$; cf. (61) below. Consequently, (59) can be written in the form $a_r{}^s = a_r$, if a_r is defined by (56). Finally, since $a_r{}^s = a_r$ holds

for all $\phi(r)$ integers s satisfying (21), it is seen from (39) that (14) becomes precisely (55).

This completes the proof of (IV). The crucial point was, of course, that, just as in (III), the $\phi(r)$ coefficients of the interior sum in (14) proved to be independent of s. Correspondingly, both (III) and (IV) can be thought of as paradigmata of (II), the *class functions* $\beta_r(n)$ of the *representations* compatible with (5) being the functions $c_r(n)$, $|\mu(r)|\, c_r(n)$ in the respective cases (III), (IV). In particular, $\beta_r(n)$ is missing in the second (but not in the first) case unless $|\mu(r)| = 1$, that is, unless r is square-free.

It may be mentioned that the elementary identity (60), used in the reduction of (59) to (56), is just the particular case of the relation

$$(61) \qquad c_m(n)/\phi(m) = \mu(m/(m,n))/\phi(m/(m,n)),$$

pointed out by Tegnér [10] for any pair of positive integers m, n and their least common divisor (m,n); cf. also Hölder [7].

6. Any function f satisfying (45) determines a unique pair of functions f', f^* by means of which f is representable in the respective forms (32), (42). Hence, (III) and (IV) supply two sufficient criteria for one and the same property, namely, the almost-periodicity (B), of an arbitrary f. In fact, the initial restriction (45), which is necessary in (IV), is quite immaterial, since, if $f(n)$ is given for every $n > 1$, the value of $f(1)$ can be chosen arbitrarily, a finite number of values $f(n)$ having no influence on the almost-periodicity (B) or on the Fourier expansion (B) of f.

It will now be shown that neither of the theorems (III), (IV) is contained in the other:

(V) *The two sufficient criteria which* (III) *and* (IV) *supply for the almost-periodicity* (B) *of a function* f *are independent, that is, condition* (37) *neither implies, nor is implied by, condition* (54).

In fact, if the values $f'(1), f'(2), \cdots$ are so chosen that (37) is satisfied and the numerical series (40) belonging to $r = 4$ has a sum distinct from 0, then, since (54) implies (58), it is clear that (37) cannot suffice for (54).

The remaining negation of (V) follows by choosing $f^*(n)$ to be 1 or 0 according as n is or is not 2. Then (54) is satisfied, since only one term of the series (54) is distinct from 0. It remains to show that this example does not satisfy (37).

First, since $f^*(2) = 1$ and $f^*(n) = 0$ for $n \neq 2$, it is seen from (42) that $f(n)$ is 1 or 0 according as n is or is not an odd number distinct from 1. Since (32) is equivalent to Möbius' inversion

$$f'(n) = \sum_{d|n} \mu(n/d) f(d) \tag{62}$$

for arbitrary mates f, f', it follows that the present $f'(n)$ is identical with the sum $\Sigma \mu(n/d)$ in which d runs through all those odd divisors of n which are distinct from 1. Hence, if n is an odd prime p, it is clear that $f'(p)$ is represented by the single term $\mu(p/d) = \mu(p/p)$; so that $f'(p) = \mu(1) = 1$. Since the sum of the reciprocal values of all primes p is divergent, it follows that (37) is violated.

This completes the proof of (V).

Due to the representation $\Sigma \mu(n/d)$ of $f'(n)$ in the last example, it is possible to conclude from the prime number theorem that, in this example, the series

$$\sum_{k=1}^{\infty} f'(k)/k \tag{63}$$

is convergent (although (37) is not satisfied). This suggests the question as to whether or not (5) *always* implies the convergence (though not, of course, the absolute convergence) of (63). The answer proves to be affirmative. In view of (I) and (IV), much more than this assertion is contained in the following theorem:

(VI) *If the function $f(n)$ is almost-periodic (B), then* (63) *is a convergent series.*

The point in (VI) is that the assumption of the mere existence of $M(f)$, an assumption weaker than the almost-periodicity (B) of f, is insufficient for the convergence of (63). This was shown by an appropriate construction ([11], p. 13), while the question as to the sufficiency of the almost-periodicity (B) of f remained there undecided (cf. [11], p. 26).

If f is almost-periodic (B), the same is true of $|f|$. Hence, the mean-value (1) exists for both functions $h = f$, $h = |f|$. This means that $M(f_1)$ and $M(f_2)$ exist for the linear combinations $f_1 = |f| + f$, $f_2 = |f| - f$ of $f, |f|$. On the other hand, it is clear from the distributive character of the connection (32) or (62) between the f and f' that, if (63) converges when f is replaced by f_i, where $i = 1, 2$, then (63) must converge for $f = \frac{1}{2}(f_1 - f_2)$ itself. But $M(f_i)$ exists and $f_i \geqq 0$ holds for $i = 1, 2$, if, without loss of generality, f is assumed to be real-valued. Hence, in order to prove (VI), it is sufficient to ascertain that the existence of $M(f)$ and the assumption $|f| \geqq 0$ imply the convergence of (63). But the truth of this implication was recently established ([12], p. 6); it depends on somewhat more than the prime number theorem (cf. [12], p. 9).

Incidentally, (VI) itself contains the prime number theorem. In fact,

$f(n)$ is 1 or 0 according as $n = 1$ or $n > 1$, then $f(n)$ clearly is almost-periodic (B), with $0 + 0 + \cdots$ as Fourier series, while (62) shows that $f'(n)$ becomes identical with $\mu(n)$. It follows therefore from (VI) that the series (63) converges when f' is Möbius' function μ. But this is known to be equivalent to the prime number theorem.

In the above proof of (VI), only the existence of $M(f)$ and $M(|f|)$ was needed. Hence, (VI) can be generalized as follows:

(VII) *If f is a function for which both $M(f)$ and $M(|f|)$ exist, then* (63) *is a convergent series.*

That (VII) *actually* is more general than (VI), follows from the existence of functions f which attain only the values 0 and 1 but fail to be almost-periodic (B), although $M(f)$ exists ([11], p. 28).

Remark. Corresponding to Möbius' inversion (62) of (32), there is an explicit inversion formula of (42), (45), namely,

$$(64) \quad f^*(n) = \sum_{d|n} f(d) - \sum_{d|n-1} f(d), \quad \text{where} \quad n > 1; \qquad f^*(1) = f(1).$$

In fact, it is clear from (42) that

$$(65) \qquad \sum_{d|n} f(d) = \sum_{k=1}^{n} f^*(k).$$

But (65) is equivalent to (64).

Incidentally, (32) shows that (65) can be written in the form

$$(66) \qquad f(n) = F'(n),$$

where

$$(67) \qquad F(n) = \sum_{k=1}^{n} f^*(k).$$

Since (62) is equivalent to (32), the explicit form of (66) is

$$(68) \qquad f(n) = \sum_{d|n} \mu(n/d) F(d).$$

Thus (42) is identical with (68) in virtue of (67).

7. In view of (III) and (IV), it seems to be worthwhile to mention an instance of a Fourier expansion (14) which fails to be of the particular form of the Ramanujan series (38) occurring in (III), (IV), although it belongs to an almost-periodic (B) function $f(n)$ of classical arithmetical significance:

(VIII) *Let $f(n)$ denote the characteristic function of the set of the*

positive integers which are sums of three squares. Then $f(n)$ is almost-periodic (B) and has the limit-periodic Fourier series (B)

$$f(n) \sim 6^{-1} + \sum_{r=1}^{\infty} 2^{-r} \sum_{1 \leqq q \leqq 2^r} \lambda_q \epsilon(nq/2^r), \tag{69}$$

where q is odd, $\epsilon(x) = \exp(2\pi i x)$ and

$$\lambda_q = -\log\{1 - \epsilon(q/8)\}, \tag{70}$$

if the imaginary part of the logarithm is chosen between $\pm\pi$.

It is understood that by the characteristic function of a set of positive integers is meant the function $f(n)$ which is 1 or 0 according as n is or is not in the set.

Let u and v be fixed positive integers. Then, since

$$\sum_{j=1}^{u} \epsilon(nj/u)$$

is u or 0 according as n is or is not divisible by u, the characteristic function of the arithmetical progression

$$u + v, \quad 2u + v, \cdots, \quad mu + v, \cdots \tag{71}$$

is represented by the trigonometric sum

$$(1/u) \sum_{j=1}^{u} \epsilon(-vj/u)\epsilon(nj/u). \tag{72}$$

According to Legendre, the sums of three squares are the integers of the form $4^{k-1}(8m-1)$, where $k = 1, 2, \cdots$ and $m = 1, 2, \cdots$. Let $f^k(n)$ denote the characteristic function of the subset which results when k is fixed and m varies. Then, if $f(n)$ is defined as at the beginning of (VIII), it is easy to prove that f is almost-periodic (B) and that, as a matter of fact, the k-th partial sum of the series $f^1 + f^2 + \cdots$ tends to f in the mean of the (B)-space, as $k \to \infty$ ([11], p. 27). Hence, corresponding to the transition from (17) to (14) in the proof of (I), the Fourier analysis (B) of f can be obtained by collecting the amplitudes which belong to a fixed frequency in the Fourier expansions of the k functions $f^1, \cdots, f^k$ and then letting $k \to \infty$. But this calculation can readily be carried out, as follows:

The set having the characteristic function f^k was defined to consist of the integers $4^{k-1}(8m-1)$, where k is fixed and $m = 1, 2, \cdots$. Clearly, this set is identical with the case $u = 2^{2k+1}$, $v = -4^{k-1}$ of the arithmetical progression (71). Hence, the Fourier series (B) of $f(n)$ can be obtained by

substituting $u = 2^{2k+1}$, $v = -4^{k-1}$ in (72) and then performing summation over all positive integers k. Consequently, the Fourier series (B) of $f(n)$ is

$$f(n) \sim \sum_{k=1}^{\infty} \sum_{j=1}^{2^{2k+1}} 2^{-2k-1}\epsilon(j/8)\epsilon(nj/2^{2k+1}). \tag{73}$$

Since $\epsilon(n) = 1$, this means that there exist certain coefficients $b_r{}^q$ satisfying

$$f(n) \sim \sum_{k=1}^{\infty} 2^{-2k-1} + \sum_{r=1}^{\infty} \sum_{1 \leqq q \leqq 2^r} b_r{}^q\, \epsilon(nq/2^r). \tag{74}$$

In order to prove (69) and (70), let r, q be two positive integers the second of which is odd and less than 2^r. Then $b_r{}^q$ in (74) is the sum of those coefficients $2^{-2k-1}\epsilon(j/8)$ of (73) for which the quotient $j/2^{2k+1}$ multiplying n has the fixed value $q/2^r$. Since $(q, 2^r) = 1$, these pairs of summation indices j, k can be parametrized by placing $j = qm$ and $2^{2k+1} = 2^r m$, where m runs through all positive integers. Consequently,

$$b_r{}^q = 2^{-r} \sum_{m=1}^{\infty} m^{-1}\epsilon(qm/8).$$

It follows therefore from (8) that (since the expansion $-\log(1 - z) = \sum_{m=1}^{\infty} m^{-1} z^m$ remains valid on $|z| = 1$ if $z \neq 1$),

$$b_r{}^q = -2^{-r} \log\{1 - \epsilon(q/8)\}, \tag{75}$$

if the determination of the logarithm continuously is derived from that of $\log(1 - z)$, where $|z| < 1$ and $\log 1 = 0$.

This completes the proof of (VIII), since (74) is identical with (69) in virtue of (75) and (70).

8. Theorems (III), (IV) and (VIII) have been deduced directly from (I), rather than from the complicated criterion (II). However, all that is responsible for the complications in (II) is the fact that the sufficient criterion supplied by (II) is necessary as well. Thus it will be convenient to isolate a *simple* particular case of (II), supplying a *sufficient* criterion which contains not only (III), (IV) and (VIII) but is also applicable, for instance, to the various sequences of Kloosterman's sums.

(IX) *Let $e_1(n), e_2(n), \cdots$ be a sequence of functions which are almost periodic (B) and have the following property: There exist constants*

$$\alpha_r{}^s, \quad \textit{where} \quad (s, r) = 1,\ 1 \leqq s \leqq r \quad \textit{and} \quad r = 1, 2, \cdots, \tag{76}$$

such that the Fourier series (B) of the functions $e_1, e_2, \cdots$ of n are expressible in terms of the sequence $\beta_1, \beta_2, \cdots$ of the trigonometric sums

$$\beta_r(n) = \sum_{1 \leqq s \leqq r}^{(s,r)=1} \epsilon(-\alpha_r{}^s + ns/r) \tag{77}$$

in the form

$$e_r(n) \sim \sum_{d|r} \beta_d(n) \tag{78}$$

(which is equivalent to

$$\beta_r(n) \sim \sum_{d|r} \mu(r/d) e_d(n), \tag{79}$$

by Möbius' inversion). Then, if $g(n)$ is any function satisfying

$$\sum_{k=1}^{\infty} M(|e_k|)|g(k)| < \infty, \tag{80}$$

the function $f(n)$ defined by the linear transformation

$$f(n) = \sum_{m=1}^{n} e_m(n) g(n) \tag{81}$$

is almost-periodic (B) and has the Fourier series (14) *of the particular type*

$$f(n) \sim \sum_{r=1}^{\infty} b_r \beta_r(n), \qquad (B), \tag{82}$$

in which the ("shifted") Fourier constants b_r are given by the absolutely convergent series

$$b_r = \sum_{r|k} g(k). \tag{83}$$

In order to see this, let (81) be identified with (2). Then (3) is satisfied by $N_m = m$. On the other hand, due to the possibility allowed in (11), (7) and (9), it is clear from the device (47) applied in the proof of (IV), that the identification of (81) in (IX) with (2) in (I) is admissible (in (47), the assignments $n < m$ and $m \leqq n$ now become $n \leqq m$ and $m < n$ respectively). Hence, if $\lambda_m(k)$ in (30) is chosen to be 1 for every m and every k, it is sufficient to go over the assumptions and assertions of (II) step by step, in order to obtain (IX).

The formula corresponding to (41) in (III) is somewhat obscured in the generalization (IX) of (III), although (83) corresponds to (40). First, it is clear from (77) that only the first of the trigonometric sums $\beta_1(n)$, $\beta_2(n), \cdots$ contains a term independent of n, and that $\beta_1(n)$ is identical with this term, that is, with the constant $\epsilon(-\alpha_1{}^1)$. It follows therefore from (82) and (1) that $M(f)$, instead of being represented by the case $r = 1$ of (83), has the value

$$M(f) = \epsilon(-\alpha_1{}^1) b_1 = \epsilon(-\alpha_1{}^1) \sum_{k=1}^{\infty} g(k). \tag{84}$$

This also explains an apparent paradox in the wording of (IX), namely, the absolute convergence of the series (83) *whenever* (80) is satisfied. In fact, if all but one of the values $g(1), g(2), \cdots$ are chosen to be 0, it follows from (81) and (84) that

$$M(e_1) = M(e_2) = \cdot\cdot\cdot = \epsilon(-\alpha_1{}^1) \tag{85}$$

(this can be seen from (78) also). But $M(|\,h\,|)$ cannot be less than $|\,M(h)\,|$, and the exponential function ϵ occurring in (85) cannot become 0.

9. All of the arithmetical algorithms (2) considered thus far led to Fourier expansions (B) which, instead of having the *general* limit-periodic form (14), induce a "class condition," expressed by the phase rule (22) or by the corresponding reduction of (14) in (I) to (82), (77) in (IX). One might expect that this will be the case for every algorithm (2) of classical arithmetical interest. But it turns out that there is at least one classical algorithm, of substantially arithmetical significance, in which (14) cannot be contracted into (82), (77). It results if the "Eratosthenian" and "Eulerian" substitutions, (32) and (42), are replaced by the algorithm

$$f(n) = \sum_{m=1}^{n} \left\{ \frac{n}{m} - \left[\frac{n}{m}\right] \right\} f^0(m), \tag{86}$$

which underlies Dirichlet's divisor problem ([3]; cf. [4]).

In (86), the bracket [ ] is the symbol of the greatest integer not exceeding the quotient n/m. Hence, if (86) is thought of as written in the form (2), where

$$g(n) = f^0(n), \tag{87}$$

then (3) is satisfied by $N_m = m - 1$ for $m = 1, 2, \cdot\cdot\cdot$. It is also seen from (86) and (2) that, while

$$f(1) = 0, \qquad f(2) = 0, \tag{88}$$

$f(n)$ will contain $f^0(n-1)$ for every $n > 2$. In other words, if $n > 2$, then $f(n)$ is of the form $f(n) = F_{n-2} + e_{n-1}(n) f^0(n-1)$, where F_{n-2} is a linear form in the $n - 2$ values $f^0(m)$ preceding $f^0(n-1)$, and the absolute constant $e_{n-1}(n)$ does not vanish. Consequently, there exists for every function $f(n)$ satisfying the initial restriction (88) a function $f^0(n)$ by means of which $f(n)$ is representable in the form (86) for every n, and f^0 is uniquely determined by f and by an arbitrary assignment of an initial value

$$f^0(1). \tag{89}$$

For every positive integer m, let $e^m = e^m(n)$ denote the non-negative function assigned by

$$(90)\qquad e^m(1) = 1/m, \quad e^m(2) = 2/m, \cdots, \\ e^m(m-1) = (m-1)/m; \quad e^m(m) = 0$$

and by the requirement that $e^m(n) = e^m(n+m)$ for every n (it is understood that, if $m = 1$, the assignment (90) gives

$$(91)\qquad e^1(n) = 0$$

for every n). In terms of these functions $e^m(n)$, define the functions $e_m(n)$ by (47). Then it is clear that (86) and (87) can be written in the form (2). Since the function e_m is non-negative and becomes identical with the periodic function e^m after the first period of e_m, it is also clear that $M(|e_m|) = M(e_m)$ exists and is identical with the arithmetical mean of the m values (90). Hence

$$M(|e_m|) = \sum_{j=1}^{m-1} j/m^2 \rightarrow \tfrac{1}{2}$$

as $m \rightarrow \infty$. Consequently, (5) is satisfied if and only if the series formed by the values (87) is absolutely convergent. This leads to the following theorem (in which the Fourier expansion (B) does *not* have the phase properties specified above):

(X) *For an arbitrary function $f = f(n)$ subject to* (88) *and for any choice of an initial constant* (89), *there exists a unique function $f^0 = f^0(n)$ by means of which f is representable in the form* (86). *Suppose that the function f^0 belonging to f satisfies the condition*

$$(92)\qquad \sum_{k=1}^{\infty} |f^0(k)| < \infty.$$

Then f is almost-periodic (B) and has the Fourier series (B)

$$(93)\qquad f(n) \sim \sum_{r=1}^{\infty} \sum_{1 \leqq s \leqq r}^{(s,r)=1} a_r{}^s \, \epsilon(ns/r),$$

in which the Fourier constants are given by the absolutely convergent series

$$(94)\qquad a_r{}^s = \sum_{r|k} \lambda_r{}^s(k) f^0(k)/k^2,$$

where the λ denote absolute constants

$$(95)\qquad \lambda_r{}^s(k) = \sum_{n=1}^{k-1} n\epsilon(-ns/r).$$

The sum (95) denotes 0 when it is vacuous, that is, when $k = 1$; cf. (91).

First, the assumption (92) of (X) was seen to be identical with the assumption (5) of (I). Next, substitution of (90) into (13) gives

$$m^2\alpha_m(j) = \sum_{n=1}^{m-1} n\epsilon(-jn/m).$$

Hence, from (15) and (87),

$$a_r{}^s = \sum_{r|k} k^{-2}f^0(k) \sum_{n=1}^{k-1} n\epsilon(-(ksn/k)/r).$$

But this can be written in the form (94), if $\lambda_r{}^s(k)$ is defined by (95).

Accordingly, (X) follows by applying (I) to the case of Dirichlet's algorithm.

10. If $1 \leqq m \leqq n$ and $n = 1, 2, \cdots$, let $r_n(m)$ denote the residue which remains when n is divided by m. In view of the description of the matrix of (86) in **9**, it is natural to raise the question as to the asymptotic behavior of the *average fluctuation* in the set

$$r_n(1),\ r_n(2), \cdots, r_n(m), \cdots, r_n(n) \tag{96}$$

of the subsequent division residues of n, as $n \to \infty$. For a fixed n, this average fluctuation is meant to be the arithmetical mean of the *absolute* deviations of the *consecutive* elements in the ordered set (96) of n non-negative integers, that is, the arithmetical mean

$$\phi_n = (1/n) \sum_{m=1}^{n} | r_n(m+1) - r_n(m) |. \tag{97}$$

The value of $r_n(m+1)$ for $m = n$ in (97) is, in the main, immaterial as $n \to \infty$; it can naturally be defined by

$$r_n(n+1) = n, \tag{98}$$

since the least non-negative residue (mod $n+1$) of n is n.

An answer to the question is contained in the following fact:

The average fluctuation of the consecutive division residues (96) *of n is of the order of* $\log n$; *in the sense that, if ϕ_n is defined by* (97),

$$0 < \liminf_{n\to\infty} \phi_n/\log n \leqq \limsup_{n\to\infty} \phi_n/\log n < \infty. \tag{99}$$

It will remain undecided whether or not this answer is of a final nature, since the approach to be followed will neither prove nor disprove the possibility that the (positive and finite) lower and upper limits coincide in (99).

As recently proved in another connection ([11], pp. 11-12),

$$\sum_{m=1}^{n} \left| \left[\frac{n}{m}\right]\frac{m}{n} - \left[\frac{n}{m+1}\right]\frac{m+1}{n} \right| > \tfrac{1}{2}\log n - \text{const.}$$

On the other hand, if $e_m(n)$ denotes, as in **9**, the coefficient of $f^0(m)$ in the n-th of the equations (86), then, since $e_m(n) = n/m - [n/m]$, it is clear that

$$me_m(n) - (m+1)e_{m+1}(n) = m[n/m] - (m+1)[n/(m+1)].$$

But it is also clear from $e_m(n) = n/m - [n/m]$ that

$$r_n(m) = me_m(n),$$

since $[n/m]$ denoted the integral part of the quotient n/m, while $r_m(n)$ was defined as the least non-negative residue of n (mod m). The last three formula lines imply the first of the inequalities (99), since ϕ_n is defined by (97).

In the direction of the remaining assertion of (99), it is clear from the last two formula lines that

$$| r_n(m) - r_n(m+1) | \leqq | [n/m] - [n/(m+1)] | m + [n/m].$$

Since $[n/(m+1)]$ is non-negative and does not exceed $[n/m]$, the sign of absolute value is superfluous on the right of this inequality. Consequently, summation with respect to m gives

$$(100) \qquad n\phi_n \leqq \sum_{m=1}^{n} [n/m]m - \sum_{m=1}^{n} [n/(m+1)]m + \sum_{m=1}^{n} [n/m],$$

ϕ_n being defined by (97). If m is replaced by $m-1$, the second of the three terms on the right of (100) appears in the form

$$-\sum_{m=2}^{n+1} [n/m](m-1),$$

which, since $[n/m](m-1)$ vanishes for $m=1$ and for $m=n+1$, is identical with

$$-\sum_{m=1}^{n} n/m(m-1), \text{ i.e., } -\sum_{m=1}^{n} [n/m]m + \sum_{m=1}^{n} [n/m].$$

Accordingly, (100) can be contracted into

$$n\phi_n \leqq 2\sum_{m=1}^{n} [n/m]; \text{ so that } \phi_n \leqq 2\sum_{m=1}^{n} 1/m,$$

since $[n/m]/n \leqq 1/m$. It follows therefore from $\sum_{m=1}^{n} 1/m \sim \log n$ that the proof of the last of the inequalities (99) is complete.

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A PARTITION FUNCTION WITH THE PRIME MODULUS $P > 3$.*

By JOHN LIVINGOOD.**

1. Recently Lehner [2][1] developed a convergent series for $p_1(n)$ and $p_2(n)$, the number of partitions of a positive integer n into summands of the form $5l \pm 1$ and $5l \pm 2$ respectively. The purpose of this paper is to develop a similar series for $p_1(n), p_2(n), \cdots, p_{(p-1)/2}(n)$, the number of partitions of a positive integer n into summands of the form $pl \pm 1$, $pl \pm 2, \cdots$, $pl \pm (p-1)/2$, p being a prime greater than 3. Here, as in the previous paper, we follow the method of Rademacher [5].

We consider the generating functions

$$F_a(x) = \prod_{l=1}^{\infty} (1 - x^l)^{-1} = \sum_{n=0}^{\infty} p_a(n) x^n \tag{1.1}$$

$$l \equiv \pm a \pmod p, \qquad (a = 1, 2, \cdots, (p-1)/2)$$

convergent within the unit circle. To determine the asymptotic behavior of $F_a(x)$ near a rational point on a circle concentric to the unit circle but interior to it, we subject x to the transformation $x \to x'$ where

$$x = \exp(2\pi i h/k - 2\pi z/k), \qquad x' = \exp(2\pi i h'/k - 2\pi/kz). \tag{1.2}$$

Here $\Re(z) > 0$, h and k are coprime integers satisfying $0 \leq h < k$, and h' is any fixed solution of

$$hh' \equiv -1 \pmod k. \tag{1.21}$$

We then derive a functional equation connecting $F_a(x)$ and $F_b(x')$ for all k divisible by p ($b = 1, 2, \cdots, (p-1)/2$ depending upon h). If $p \nmid k$, we write

$$x'' = \exp(2\pi i H'/k - 2\pi/Kz), \qquad HH' \equiv -1 \pmod k \tag{1.3}$$

and find a similar relation between $F_a(x)$ and certain new functions $H_a(x'')$. Certain sums of roots of unity appear. The application of the Rademacher method requires that these sums be reduced to incomplete Kloosterman sums, so that a better estimate than $O(k)$ will result.

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** The author wishes to express his appreciation to Professor Hans Rademacher for suggesting this problem and for giving much helpful advice in its solution.

[1] Square bracket numbers refer to the bibliography.

We now introduce the Cauchy integral and apply the transformation equations (3.93) and (3.94). If $p|k$ we obtain

$$p_a^{(1)}(n) = \sum_{b=1}^{(p-1)/2} \sum_{h,k}{}' \omega_a(h,k)\exp(-2\pi ihn/k)$$

$$(1.4) \qquad \times \int_{-\theta'}^{\theta''} \sum_{\nu=0}^{\infty} p_b(\nu)\exp(2\pi ih'\nu/k)$$

$$\times \exp\{-(\pi/k^2w)(2\nu - B/6p) + \pi w(2n - A/6p)\}d\phi,$$

$$0 \leq h < k \leq N, \quad \theta' = \theta'_{h,k}, \quad \theta'' = \theta''_{h,k}, \quad B = p^2 - 6pb + 6b^2.$$

Next the integrand is split into two parts, depending upon the sign of the coefficient of w^{-1}. We note that this coefficient is positive for $\nu = 0, 1, \cdots, [B/12p]$. For certain p, $\nu = 0$ is the only admissible value. This is the result that Lehner obtained for $p = 5$. It is also true for all $p \leq 17$. However, for other p, there may be more admissible values for ν. For example, if $p = 19$, $\nu = 0$ or 1. This interesting information results in the extension of the previous work. The contributions furnished by the case $p|k$ when the coefficient of w^{-1} is negative, and by the case $p \nmid k$, follow as in the special case $p = 5$.

Finally we obtain asymptotic ratios for $p_1(n), p_2(n), \cdots, p_{(p-1)/2}(n)$.

I. The Transformation Equations.

2. The first object is to derive a transformation equation for $F_a(x)$ in (1.1). Consider the case $p|k$. $F_a(x)$ is regular, without zeros, in the unit circle; therefore $\log F_a(x)$ is single-valued in the same region if we choose a specific branch of the logarithm, for example, the one given by $\log F(0) = 0$. Then

$$(2.1) \qquad G_a(x) = \log F_a(x) = -\sum_{l=1}^{\infty} \log(1 - x^l) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} (x^{lm}/m),$$

$$l \equiv \pm a \pmod{p}.$$

Placing

$$x = \exp(2\pi ih/k) \times X, \quad X = \exp(-y), \quad \Re(y) > 0$$

$$(2.2) \qquad m = qk + \mu, \quad 1 \leq \mu \leq k; \qquad q, r = 0, 1, 2, \cdots$$

$$l = rk + \lambda, \quad 1 \leq \lambda < k, \quad \lambda \equiv \pm a \pmod{p}$$

in (2.1) we have

$$(2.21) \quad G_a(x) = \sum_{\lambda} \sum_{\mu} \exp(2\pi ih\mu\lambda/k) \sum_{r,q} \frac{1}{qk + \mu} \exp\{-(rk + \lambda)(qk + \mu)y\}$$

where λ, μ, r, and q assume the values described above.[2]

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[2] Throughout this paragraph, λ, μ, q and r assume these values.

Application of the Mellin formula to (2.21) gives

$$G_a(x) = \sum_\lambda \sum_\mu \exp(2\pi i h\mu\lambda/k) \sum_{r,q} (1/2\pi i) \int_{(a)} \frac{\Gamma(s)}{qk+\mu} \frac{ds}{(rk+\lambda)^s(qk+\mu)^s y^s}$$
$$= \sum_\lambda \sum_\mu \exp(2\pi i h\mu\lambda/k)\,(1/2\pi i) \int_{(a)} \frac{\Gamma(s)}{y^s k^{2s+1}} \sum_{r=0}^{\infty} (r+\lambda/k)^{-s} \sum_{q=0}^{\infty} (q+\mu/k)^{-s-1}$$

$$G_a(x) = \frac{1}{2\pi i k} \sum_\lambda \sum_\mu \exp(2\pi i h\mu\lambda/k) \int_{(a)} \frac{\Gamma(s)}{y^s k^{2s}} \zeta(s,\lambda/k)\zeta(1+s,\mu/k)\,ds \tag{2.3}$$

where

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \qquad 0 < a \leq 1$$

is the Hurwitz Zeta function, and where the (α) under the integral sign means that the path of integration is from $\alpha - i\infty$ to $\alpha + i\infty$.

Recall the Zeta functional equation

$$\begin{aligned} \zeta(s,\lambda/k) = \Gamma(1-s)\, \frac{2}{(2\pi k)^{1-s}} \{&\sin(\pi s/2) \sum_{\nu=1}^{k} \cos 2\pi\lambda\nu/k \cdot \zeta(1-s,\nu/k) \\ &+ \cos(\pi s/2) \sum_{\nu=1}^{k} \sin 2\pi\lambda\nu/k \cdot \zeta(1-s,\nu/k)\}. \end{aligned} \tag{2.4}$$

Applying (2.4), replacing[3] $\exp(2\pi i h\mu\lambda/k)$ by $\cos(2\pi h\mu\lambda/k)$ and $\exp(2\pi i h\mu\lambda/k)$ by $i\sin(2\pi h\mu\lambda/k)$ in the first and second terms of the right member of (2.3), and setting $z = (yk/2\pi)$, we have, after simplification

$$\begin{aligned} G_a(x) = {}& \frac{1}{4\pi i k^2} \sum_\lambda \sum_{\mu,\nu} \cos 2\pi h\mu\lambda/k \cos 2\pi\lambda\nu/k \\ &\times \int_{(a)} \frac{\zeta(1+s,\mu/k)\zeta(1-s,\nu/k)}{z^s \cos \pi s/2}\, ds \\ &+ \frac{1}{4\pi k^2} \sum_\lambda \sum_{\mu,\nu} \sin 2\pi h\mu\lambda/k \sin 2\pi\lambda\nu/k \\ &\times \int_{(a)} \frac{\zeta(1+s,\mu/k)\zeta(1-s,\nu/k)}{z^s \sin \pi s/2}\, ds \end{aligned} \tag{2.41}$$

where $z^s = \exp(s \log z)$ and where we can take $|\mathfrak{J} \log z| < \pi/2$ since $\mathfrak{R}(z) > 0$.

We now define ρ by setting

$$\lambda \equiv h'\rho \pmod k, \qquad 0 < \rho < k \tag{2.5}$$

where h' is given by (1.21). This, together with (1.21), shows that

$$\rho \equiv -h\lambda \pmod k. \tag{2.51}$$

---

[3] This is permissible since we sum over λ and since $\lambda \equiv \pm a \pmod p$.

In addition, since $p|k$, (1. 21) implies $hh' \equiv -1 \pmod{p}$ and hence, by Fermat's Theorem,

$$h' \equiv -h^{p-2} \pmod{p}.$$

Since $p|k$ and $\lambda \equiv \pm a \pmod{p}$, (2. 51) implies

$$\rho \equiv \pm ha \equiv \pm b \pmod{p}. \tag{2.52}$$

Introducing the new summation letter ρ and replacing s by $-s$, we have

$$\begin{aligned} G_a(x) = {} & (1/4\pi ik^2) \sum_{\rho} \sum_{\mu,\nu} \cos 2\pi\mu\rho/k \cos 2\pi h'\rho\nu/k \\ & \times \int_{(-a)} \frac{\zeta(1-s,\mu/k)\zeta(1+s,\nu/k)}{z^{-s}\cos \pi s/2}\, ds \\ & + (1/4\pi k^2) \sum_{\rho} \sum_{\mu,\nu} \sin 2\pi\mu\rho/k \sin 2\pi h'\rho\nu/k \\ & \times \int_{(-a)} \frac{\zeta(1-s,\mu/k)\zeta(1+s,\nu/k)}{z^{-s}\sin \pi s/2}\, ds \end{aligned} \tag{2.6}$$

where the restrictions on ρ are given by (2. 5) and (2. 52).

The appearance of $\cos \pi s/2$ and $\sin \pi s/2$ in the denominators enables us to shift the path of integration from $(-\alpha)$ to (α), taking into consideration the residues at the various poles. Doing this and comparing the result with (2. 41), we find two expressions of the same form, except that we now have ρ instead of λ, h' instead of h, μ and ν interchanged, and z^{-1} instead of z. Hence, using the definition (1. 2) of x', we obtain

$$G_a(x) = G_b(x') - 2\pi i(R_1 + R_2) \tag{2.7}$$

where R_1 and R_2 are the sums of the residues of the first and second terms respectively in the right member of (2. 6) with the sign of integration moved to the extreme left in each case.

These residues turn out to be the following:[4]

$$R_1 = Az/12ikp - B/12ikpz; \tag{2.8}$$

$$R_2 = -\tfrac{1}{2} \sum_{\lambda=1}^{k-1} ((\lambda/k))((h\lambda/k)), \qquad \lambda \equiv \pm a \pmod{p}$$

where

$$A = p^2 - 6pa + 6a^2, \qquad B = p^2 - 6pb + 6b^2, \tag{2.81}$$
$$((x)) = x - [x] - \tfrac{1}{2} + \tfrac{1}{2}\delta(x).$$

Thus, by use of (2. 7) and (2. 8) and by taking exponentials, we obtain the desired transformation equation for the case $p|k$:

[4] Calculation of the residues follows as in Lehner's paper.

$$F_a(x) = \omega_a(h, k) \exp\{(\pi/6kp)(B/z - Az)\}F_b(x'). \tag{2.9}$$

Here x and x' are defined by (1.2), b by (2.52), A and B by (2.81) and

$$\omega_a(h, k) = \exp\{\pi i \sigma_a(h, k)\} \tag{2.91}$$

with

$$\sigma_a(h, k) = \sum_{\lambda=1}^{k-1} ((\lambda/k))((h\lambda/k)), \qquad \lambda \equiv \pm a \pmod p. \tag{2.92}$$

3. Now we consider the case $p \nmid k$. Here $F_a(x)$ does not transform into $F_b(x')$; however, $F_a(x)$ will be transformed into a certain new function by a method similar to the preceding one. Choose x'', H, H', K as in (1.3) and place

$$\begin{aligned} &x = \exp(2\pi i h/k) \times X, \quad && X = \exp(-y), \quad && \Re(y) > 0 \\ &m = qk + \mu, && 1 \leq \mu \leq k; && r, q = 0, 1, 2, \cdots \\ &l = rK + \lambda, && 1 \leq \lambda < K, && \lambda \equiv \pm a \pmod p. \end{aligned} \tag{3.1}$$

As in **2**, we find [5]

$$\begin{aligned} G_a(x) = (1/4\pi i k K) &\sum_\lambda \sum_\mu \sum_\nu \cos 2\pi h \mu\lambda/k \cos 2\pi\lambda\nu/K \\ &\times \int_{(a)} \frac{\zeta(1+s, \mu/k)\zeta(1-s, \nu/K)}{z^s \cos \pi s/2}\, ds \\ + (1/4\pi k K) &\sum_\lambda \sum_\mu \sum_\nu \sin 2\pi h\mu\lambda/k \sin 2\pi\lambda\nu/K \\ &\times \int_{(a)} \frac{\zeta(1+s, \mu/k)\zeta(1-s, \nu/K)}{z^s \sin \pi s/2}\, ds; \\ &(\nu = 1, 2, \cdots, K). \end{aligned} \tag{3.2}$$

Define λ^* by setting

$$\lambda \equiv pH'\lambda^* \pmod k, \qquad 0 < \lambda^* \leq k \tag{3.3}$$

where H' is any fixed solution of $HH' \equiv -1 \pmod k$ and $H = ph$. We observe that (3.3) implies $\lambda^* \equiv -h\lambda \pmod k$. Introducing λ^*, replacing s by $-s$, and shifting the path of integration as in **2**, we obtain

$$\begin{aligned} G_a(x) = (1/4\pi i k K) &\sum_\lambda \sum_\mu \sum_\nu \cos 2\pi\mu\lambda^*/k \cos 2\pi\lambda\nu/K \\ &\times \int_{(a)} \frac{\zeta(1-s, \mu/k)\zeta(1+s, \nu/K)}{z^{-s} \cos \pi s/2}\, ds \\ + (1/4\pi k K) &\sum_\lambda \sum_\mu \sum_\nu \sin 2\pi\mu\lambda^*/k \sin 2\pi\lambda\nu/K \\ &\times \int_{(a)} \frac{\zeta(1-s, \mu/k)\zeta(1+s, \nu/K)}{z^{-s} \sin \pi s/2}\, ds \\ &- 2\pi i(R_1 + R_2) = L_a(x) - 2\pi i(R_1 + R_2). \end{aligned} \tag{3.4}$$

---

[5] Throughout this paragraph, λ and μ assume the values given in (3.1) and ν assumes the values $1, 2, \cdots, K$.

In order to obtain the new function into which $F_a(x)$ is transformed, we now retrace our steps from (3.4). Replace $\cos(2\pi\lambda\nu/K)$ by $\exp(2\pi i\lambda\nu/K)$ and $\sin(2\pi\lambda\nu/K)$ by $i^{-1}\exp(2\pi i\lambda\nu/K)$ and apply the transformation equation for $\zeta(s,\lambda^*/k)$. After simplification, we find

$$L_a(x) = (1/2\pi iK)\sum_\lambda\sum_\nu \exp(2\pi i\lambda\nu/K) \times \int_{(a)} (z^s\Gamma(s)/(2\pi k)^s)\zeta(s,\lambda^*/k)\zeta(1+s,\nu/K)\,ds.$$

Application of the Mellin formula gives

$$\begin{aligned} L_a(x) &= \sum_\lambda\sum_\nu \exp(2\pi i\lambda\nu/K)\sum_{r,q=0}^{\infty}(rK+\nu)^{-1}\exp\{-(2\pi/Kz)(qk+\lambda^*)(rK+\nu)\} \\ &= -\sum_\lambda\sum_q \log\{1-\exp[-(2\pi/Kz)(qk+\lambda^*)+2\pi i\lambda/K]\}. \end{aligned} \tag{3.5}$$

Examination of (3.1) reveals that since $p \nmid k$, λ runs over a complete residue system modulo k twice in some order. We therefore define α by

$$k\alpha \equiv a \pmod p, \qquad 0 < \alpha < p. \tag{3.6}$$

This, along with (3.3), yields

$$\lambda \equiv pH'\lambda^* \pm k\alpha \pmod K. \tag{3.61}$$

Inserting (3.61) into (3.5), we obtain

$$L_a(x) = -\sum_\lambda\sum_q \log\{1-\exp[-(2\pi/Kz)(qk+\lambda^*) + 2\pi iH'(qk+\lambda^*)/k \pm 2\pi i\alpha/p]\}.$$

Place

$$qk+\lambda^* = l,\ \exp(2\pi i\alpha/p) = \eta(\alpha),\ \exp(-2\pi i\alpha/p) = \overline{\eta(\alpha)}. \tag{3.71}$$

Using these notations and (1.3), we find

$$L_a(x) = -\sum_{l=1}^{\infty}\log(1-\eta(\alpha)x''^l) - \sum_{l=1}^{\infty}\log(1-\overline{\eta(\alpha)}x''^l)$$

$$L_a(x) = \log\prod_{l=1}^{\infty}(1-\eta(\alpha)x''^l)^{-1}\prod_{l=1}^{\infty}(1-\overline{\eta(\alpha)}x''^l)^{-1} = \log H_a(x'') \tag{3.72}$$

where

$$H_a(x) = \prod_{l=1}^{\infty}(1-\eta(\alpha)x^l)^{-1}\prod_{l=1}^{\infty}(1-\overline{\eta(\alpha)}x^l)^{-1} = 1+\sum_{\nu=1}^{\infty}C_a(\nu)x^\nu. \tag{3.73}$$

We now calculate the residues as in Lehner's paper, and find

$$\begin{aligned} R_1 &= Az/12ikp - 1/12ikpz + \tfrac{1}{2}\log(2\sin\pi\alpha/p); \\ R_2 &= -\tfrac{1}{2}\sum_{\lambda=1}^{K-1}((h\lambda/k))((\lambda/K)), \qquad \lambda \equiv \pm a \pmod p \end{aligned} \tag{3.8}$$

where A and $((x))$ are given by (2.81). Therefore, by use of (3.4), (3.72) and (3.8) and by taking exponentials, we obtain the desired transformation equation for the case $p \nmid k$:

$$(3.9) \quad F_a(x) = \tfrac{1}{2}\chi_a(h, k) \csc \pi\alpha/p \exp \{(\pi/6kp)(1/z - Az)\} H_a(x'')$$

where A is given by (2.81), $H_a(x)$ by (3.73), x by (1.2), x'' by (1.3) and

$$(3.91) \qquad \chi_a(h, k) = \exp \{\pi i t_a(h, k)\}$$

with

$$(3.92) \quad t_a(h, k) = \sum_{\lambda=1}^{K-1} ((h\lambda/k))((\lambda/K)), \qquad \lambda \equiv \pm a \pmod{p}.$$

From (2.9) and (3.9) we have the following result:

THEOREM 1. *The function $F_a(x)$, defined in* (1.1), *satisfies the transformation equation*

$$(3.93) \quad F_a(\exp [2\pi i h/k - 2\pi z/k]) \\ = \omega_a(h, k) \exp [(\pi/6kp)(B/z - Az)] F_b(\exp [2\pi i h'/k - 2\pi/kz])$$

for the case $p \mid k$, and the equation

$$(3.94) \quad F_a(\exp [2\pi i h/k - 2\pi z/k]) \\ = \tfrac{1}{2}\chi_a(h, k) \csc \pi\alpha/p \exp [(\pi/6kp)(1/z - Az)] H_a(exp\ [2\pi i H'/k - 2z$$

for the case $p \nmid k$.

II. The Kloosterman Sums.

4. When the transformation equations (3.93) and (3.94) are inserted into the Cauchy integral, certain sums of the roots of unity $\omega_a(h, k)$ and $\chi_a(h, k)$ appear. The estimate $O(k)$ for these sums is not sufficient for the application of Rademacher's method. Thus, we now reduce these sums to incomplete Kloosterman sums, subject to the estimate $O(n^{1/3}k^{2/3+\epsilon})$, according to Salié.

Consider the case $p \mid k$. From (2.92)

$$\sigma_a(h, k) = \sum_{\lambda=1}^{k-1} ((\lambda/k))((h\lambda/k)), \qquad \lambda \equiv \pm a \pmod{p}.$$

However, by the same procedure as that used by Lehner,[6] we are able to replace this by

$$(4.1) \quad \sigma_a(h, k) = 2 \sum_{\lambda=1}^{k-1} (\lambda/k)((h\lambda/k)) - (b/p - \tfrac{1}{2}), \qquad \lambda \equiv a \pmod{p}$$

where b is given by (2.52). Employing the definition of $((x))$ we obtain

[6] See [2], pages 644-645.

$$\sigma_a(h,k) = (2h/k^2)\sum_\lambda \lambda^2 - (2/k)\sum_\lambda \lambda[h\lambda/k] - (1/k)\sum_\lambda \lambda - (b/p - \tfrac{1}{2}),$$
$$\lambda \equiv a \pmod p$$

which upon simplification becomes

$$(4.2)\qquad \begin{aligned} 6pk\sigma_a(h,k) &= 2h\{2k^2 + 3k(2a-p) + A\} \\ &\quad - 3k(k - 2p + 2a + 2b) - 12p\sum_\lambda \lambda[h\lambda/k], \\ &\qquad \lambda \equiv a \pmod p \end{aligned}$$

where A is given by (2.81). Examination of (4.2) reveals that $6pk\sigma_a(h,k)$ is always integral. Also,

$$(4.21)\qquad \begin{aligned} 6pk\sigma_a(h,k) &\equiv 0 \pmod 3 \text{ if } 3\nmid k, \\ 6pk\sigma_a(h,k) &\equiv 2a + 2b + 2p - 3 \pmod 4 \text{ if } k \text{ is odd.} \end{aligned}$$

Next we shall evaluate a certain sum in two different ways,[7] and obtain a result containing $\sigma_a(h,k)$, from which we can find the residues of $6pk\sigma_a(h,k)$ for moduli which are multiples of k. Now

$$(4.31)\qquad \sum_\lambda ((h\lambda/k))^2 = \sum_\rho ((\rho/k))^2 = \sum_\rho (\rho/k - \tfrac{1}{2})^2 = (B - k^2)/6pk + \tfrac{1}{4}\sum_\lambda 1;$$
$$\lambda \equiv a \pmod p, \quad \rho \equiv -b \pmod p.$$

Also

$$(4.32)\qquad \begin{aligned} \sum_\lambda ((h\lambda/k))^2 &= \sum_\lambda (h\lambda/k - [h\lambda/k] - \tfrac{1}{2})^2 \\ &= h\sigma_a(h,k) + h(b/p - \tfrac{1}{2}) - (h^2/6pk)\{2k^2 + 3k(2a-p) + A\} \\ &\quad + \sum_\lambda [h\lambda/k]([h\lambda/k] + 1) + \tfrac{1}{4}\sum_\lambda 1; \qquad \lambda \equiv a \pmod p \end{aligned}$$

by the use of (4.1). Comparing (4.31) and (4.32) we find

$$(4.4)\qquad \begin{aligned} 6phk\sigma_a(h,k) &= h^2\{2k^2 + 3k(2a-p) + A\} - (k^2 - B) \\ &\quad - 3hk(2b - p) - 12pkT \end{aligned}$$

where

$$T = \tfrac{1}{2}\sum_\lambda [h\lambda/k]([h\lambda/k] + 1), \qquad \lambda \equiv a \pmod p$$

is always integral. Let $12p = fG$, where f is the greatest divisor of $12p$ which is prime to k. We obtain the following results:

$$(4.5)\qquad \begin{array}{llll} (k,12p) = p, & f = 12, G = p & (k,12p) = 3p, & f = 4, G = 3p \\ \left.\begin{array}{l}(k,12p) = 2p \\ (k,12p) = 4p\end{array}\right\}, & f = 3, G = 4p & \left.\begin{array}{l}(k,12p) = 6p \\ (k,12p) = 12p\end{array}\right\}, & f = 1, G = 12p. \end{array}$$

Denote by h' a solution of $hh' \equiv -1 \pmod{Gk}$. Since all primes in $G \mid k$, this congruence always has solutions because $(h,k) = 1$ implies $(h, Gk) = 1$.

Multiplying (4.4) by $-h'$ and remembering that $Gk \mid 2k^2$, we obtain

[7] See [6], proof of Theorem 3.

$$(4.61)\qquad 6pk\sigma_a(h,k) \equiv hu - h'v - 3k(2b-p) \pmod{Gk}$$

where

$$u = 3k(2a-p) + A, \qquad v = B - k^2.$$

From (4.21) and (4.5) we have

$$(4.62)\qquad 6pk\sigma_a(h,k) \equiv 6a + 6b + 6p - 3 \pmod{f}.$$

Set

$$(4.7)\qquad f\phi \equiv 1 \pmod{Gk}, \qquad Gk\Gamma \equiv 1 \pmod{f}.$$

Then, from (4.61) and (4.62) we find

$$(4.8)\qquad \begin{aligned} 6pk\sigma_a(h,k) &\equiv f\phi\{hu - h'v - 3k(2b-p)\} \\ &\quad + Gk\Gamma(6a+6b+6p-3) \pmod{12kp}. \end{aligned}$$

Thus, finally,

$$(4.9)\qquad \begin{aligned} \omega_a(h,k) &= \exp\,\{\pi i\sigma_a(h,k)\} = \exp\,\{(2\pi i/12kp)6pk\sigma_a(h,k)\} \\ &= \exp\,\{2\pi i[(\phi/Gk)(hu-h'v) - (3\phi/G)(2b-p) \\ &\quad + (\Gamma/f)(6a+6b+6p-3)]\}. \end{aligned}$$

5. We now consider the case $p \nmid k$. From (3.92) and an argument similar to that of Lehner's, we have

$$(5.1)\qquad \begin{aligned} t_a(h,k) &= 2\sum_{\lambda=1}^{K-1}(\lambda/K)((h\lambda/k)) = 2\sum_{\lambda=1}^{K-1}(\lambda/K)(h\lambda/k - [h\lambda/k] \\ &\quad - \tfrac{1}{2} + \tfrac{1}{2}\delta(h\lambda/k)),\ \lambda \equiv a \pmod{p}. \end{aligned}$$

As in **4**, we obtain, upon simplification,

$$(5.2)\qquad \begin{aligned} 6pkt_a(h,k) &= 2h\{2K^2 + 3K(2a-p) + A\} \\ &\quad - 3k(K - p + 2a - 2\alpha) - 12\sum_\lambda \lambda[h\lambda/k], \\ &\qquad \lambda \equiv a \pmod{p} \end{aligned}$$

where α is given by (3.6). Examination of (5.2) shows

$$(5.3)\qquad \begin{aligned} 6pkt_a(h,k) &\equiv 0 \pmod{3} \text{ if } 3 \nmid k \\ 6pkt_a(h,k) &\equiv (p - pk + 2a + 2\alpha) \pmod{4} \text{ if } k \text{ is odd} \\ 6pkt_a(h,k) &\equiv 0 \pmod{p}. \end{aligned}$$

As in **4**, we evaluate $\sum_\lambda ((h\lambda/k))^2$, $\lambda \equiv a \pmod{p}$, $\lambda = 1, 2, \cdots, K-1$ in two different ways, and upon comparing results find

$$(5.4)\qquad \begin{aligned} 6pkt_a(h,k) &= h^2\{2K^2 + 3K(2a-p) + A\} \\ &\quad - (k^2 - 1) + 6hk\alpha - 12kT' \end{aligned}$$

where

$$T' = \tfrac{1}{2}\sum_\lambda [h\lambda/k]([h\lambda/k] + 1), \qquad \lambda \equiv a \pmod{p}.$$

Let $12p = Fg$, where F is the greatest divisor of $12p$ which is prime to k. Then

$$
(5.5)\qquad
\begin{array}{llll}
(k, 12p) = 1, & F = 12p, g = 1 & (k, 12p) = 3, & F = 4p, g = 3 \\
\left.\begin{array}{l}(k, 12p) = 2 \\ (k, 12p) = 4\end{array}\right\}, & F = 3p, g = 4 & \left.\begin{array}{l}(k, 12p) = 6 \\ (k, 12p) = 12\end{array}\right\}, & F = p, \quad g = 12.
\end{array}
$$

Denote by h' a solution of $hh' \equiv -1 \pmod{gk}$. Multiplying (5.4) by $-h'$ and remembering that $g|12$ and $g|2k$, we obtain

$$(5.61)\qquad 6pkt_a(h, k) \equiv hu - h'v + 6k\alpha \pmod{gk}$$

where

$$u = 3K(2a - p) + A, \qquad v = 1 - k^2.$$

From (5.3) and (5.5)

$$(5.62)\qquad 6pkt_a(h, k) \equiv 9p^2(2a + 2\alpha + p - pk) \pmod{F}.$$

Setting

$$(5.7)\qquad F\Phi \equiv 1 \pmod{gk}, \qquad gk\gamma \equiv 1 \pmod{F}$$

and combining (5.61) and (5.62), we find

$$(5.8)\qquad \begin{aligned} 6pkt_a(h, k) \equiv{}& F\Phi(hu - h'v + 6k\alpha) \\ &+ 9gk\gamma p^2(2a + 2\alpha + p - pk) \pmod{12pk}. \end{aligned}$$

Therefore

$$\begin{aligned} \chi_a(h, k) &= \exp\{\pi i t_a(h, k)\} = \exp\{(2\pi i/12pk)6pkt_a(h, k)\} \\ &= \exp\{2\pi i[(\Phi/gk)(hu - h'v) + 6\Phi\alpha/g + (9p^2\gamma/F)(2a + 2\alpha + p - pk)]\}. \end{aligned}$$

By choosing $pH' \equiv h' \pmod{gk}$, and using $G = pg$, $H = ph$ we obtain finally

$$(5.9)\qquad \begin{aligned} \chi_a(h, k) = \exp\{&2\pi i[(\Phi/Gk)(Hu - p^2H'v) \\ &+ 6\Phi\alpha/g + (9p^2\gamma/F)(2a + 2\alpha + p - pk)]\}. \end{aligned}$$

6. Consider the sum

$$(6.1)\qquad \begin{aligned} &A(n, \nu; k; a, b; \sigma_1, \sigma_2) = \tau = \sum_h{}' \omega_a(h, k) \exp(-2\pi i(hn - h'\nu)/k) \\ &h \bmod k,\ h \equiv b\beta \pmod{p};\ \sigma_1 \leq h' < \sigma_2;\ 0 \leq \sigma_1 < \sigma_2 \leq k;\ p|k. \end{aligned}$$

Here Σ' means the sum over all integers prime to the modulus of the system, and h' denotes any solution of $hh' \equiv -1 \pmod{k}$. Examination of (2.81) shows $\omega_a(h, k)$ to be of period k. Because of the periodicity modulo k, we can change the modulus to Gk and select that h' which satisfies $hh' \equiv -1 \pmod{Gk}$. By inserting the value (4.9) for $\omega_a(h, k)$ and removing the restrictions on h and h', we find that $\tau = O(n^{1/3}k^{2/3+\epsilon})$. Hence, we secure the following result: [8]

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[8] For details of this development, see [2], pages 649-651.

THEOREM 2. *The sum*

$$(6.2)\quad \begin{aligned} &A(n,\nu;k;a,b;\sigma_1,\sigma_2)=\sum_h{}' \omega_a(h,k)\exp(-2\pi i(hn-h'\nu)/k)\\ &h \bmod k,\quad h\equiv b\beta \pmod p,\quad \sigma_1\leq h'<\sigma_2\end{aligned}$$

in which the parameters are all integers, $n>0$, $k>0$, $p\,|\,k$, $0\leq\sigma_1<\sigma_2\leq k$, *and* $a=1,2,\cdots,(p-1)/2$ *is subject to the estimate* $O(n^{1/3}k^{2/3+\epsilon})$ *uniformly in* ν, a, b, σ_1, σ_2.

By a similar method, and by use of the equations $G=pg$, $H=ph$, $HH'\equiv-1 \pmod k$, we obtain our next theorem.

THEOREM 3. *The sum*

$$(6.3)\quad \begin{aligned} &B(n,\nu;k;\sigma_1,\sigma_2;a)=\sum_h{}' \chi_a(h,k)\exp(-2\pi i(hn-H'\nu)/k)\\ &h \bmod k,\quad HH'\equiv-1 \pmod k,\quad \sigma_1\leq h'<\sigma_2\end{aligned}$$

has the estimate $O(n^{1/3}k^{2/3+\epsilon})$ *uniformly in* ν, σ_1, σ_2, a.

III. A Convergent Series for $p_a(n)$.

7. We now apply the Hardy-Littlewood method, with the sharper estimate of Kloosterman and Rademacher,[9] to the generating functions $F_a(x)$. We obtain

$$p_a(n)=(1/2\pi i)\int_C (F_a(x)/x^{n+1})\,dx$$

$$(7.1)\quad =\sum_{h,k}{}' \exp(-2\pi ihn/k)\int_{-\theta'}^{\theta''} F_a(\exp[2\pi ih/k-2\pi(z/k)])\exp(2\pi nw)$$

$$0\leq h<k\leq N,\qquad \theta'=\theta'_{h,k},\qquad \theta''=\theta''_{h,k}.$$

We now apply the transformation formulas, and write

$$(7.21)\qquad p_a(n)=p_a^{(1)}(n)+p_a^{(2)}(n)$$

where $p_a^{(1)}(n)$ is the sum of all terms for which $p\,|\,k$ and $p_a^{(2)}(n)$ is the same for $p\nmid k$. Choose β such that $a\beta\equiv+1 \pmod p$. This together with (2.52) shows that

$$h\equiv b\beta \pmod p.$$

Then we obtain from (7.1) and the remarks above

[9] See [5]; the notations here are those of Rademacher.

$$p_a^{(1)}(n) = \sum_{b=1}^{(p-1)/2} \sum_{h,k}{}' \omega_a(h,k) \exp(-2\pi i(hn/k))$$
$$(7.22) \quad \times \int_{-\theta'}^{\theta''} \sum_{\nu=0}^{\infty} p_b(\nu) \exp(2\pi i h'\nu/k) \exp\{-(\pi/k^2 w)(2\nu - (B/6p)) + \pi w(2n - (A/6p))\} d\phi,$$
$$0 \leq h < k \leq N, \qquad h \equiv b\beta \pmod p, \qquad \theta' = \theta'_{h,k}, \qquad \theta'' = \theta''_{h,k}.$$

The sum over ν is now split into two parts, $Q(n)$ and $R(n)$ depending upon B and *a fortiori* b such that in $Q(n)$ the coefficient of w^{-1} is positive and in $R(n)$ the coefficient of w^{-1} is negative. $R(n)$ contributes nothing to the final result; in fact,

$$(7.3) \qquad R(n) = O(\exp(c\pi N^{-2}) n^{1/3} N^{-1/3+\epsilon}).$$

Referring to (7.21) we note that $Q(n)$ contains all ν such that $\nu \leq [B/12p]$. If B is negative, the sum over ν is vacuous. For positive B and $\nu = 0, 1, \cdots, [B/12p]$

$$p_b(\nu) = \begin{cases} 0 \text{ if } b \nmid \nu \\ 1 \text{ if } b \mid \nu. \end{cases}$$

Here $p_b(\nu)$ denotes the number of partitions of ν into parts congruent to $\pm b \pmod p$ and since

$$\nu < B/12p = \tfrac{1}{2} \frac{p^2 - 6pb + 6b^2}{6p} = \tfrac{1}{2}(p/6 - b + b^2/p) < p/12$$

those parts that must be congruent to $\pm b \pmod p$ must be b itself, for $p - b$ or $p + b$ is too large $(p - b > p/2 > p/12 > \nu)$. Also, $B \geq 0$ means $b^2 - pb + p^2/6 \geq 0$, i. e., $b^2 - pb + p^2/4 \geq p^2/12$. Thus, the only permissible b for $Q(n)$ are $b = 1, 2, \cdots, [p(3 - \sqrt{3})/6]$. Setting $\nu = b\rho$, we find

$$Q(n) = \sum_{b=1}^{[p(3-\sqrt{3})/6]} \sum_{h,k}{}' \omega_a(h,k) \exp(-2\pi i(hn/k)) \sum_{\rho=0}^{[B/12pb]} \exp(2\pi i h' b\rho/k)$$
$$(7.4) \quad \times \int_{-\theta'}^{\theta''} \exp\{-(\pi/k^2 w)(2b\rho - (B/6p)) + \pi w(2n - (A/6p))\} d\phi,$$
$$0 \leq h < k \leq N, \qquad h \equiv b\beta \pmod p.$$

Evaluating $Q(n)$, we find

$$(7.5) \quad Q(n) = 2\pi \sum_{k=1}^{N} \sum_{b=1}^{[p(3-\sqrt{3})/6]} \sum_{\rho=0}^{[B/12pb]} A_{k,b}(n) L_k(n) + O(\exp(c\pi N^{-2}) n^{1/3} N^{-1/3+\epsilon})$$

where

$$A_{k,b}(n) = A(n, \nu; k; a, b; 0, k) \tag{7.51}$$

and

$$\begin{aligned} L_k(n) &= \frac{(B-12pb_\rho)^{1/2}}{k(12pn-A)^{1/2}} I_1\left\{\frac{\pi(12pn-A)^{1/2}(B-12pb_\rho)^{1/2}}{3pk}\right\} \quad \text{for} \quad n > A/12p \\ &= \frac{(B-12pb_\rho)^{1/2}}{k(A-12pn)^{1/2}} J_1\left\{\frac{\pi(A-12pn)^{1/2}(B-12pb_\rho)^{1/2}}{3pk}\right\} \quad \text{for} \quad n < A/12p. \end{aligned} \tag{7.52}$$

Now

$$\begin{aligned} p_a^{(2)}(n) &= \tfrac{1}{2}\sum_{h,k}{}' \chi_a(h,k) \csc \pi\alpha/p \exp(-2\pi i(hn/k)) \\ &\times \int_{-\theta'}^{\theta''} \sum_{\nu=0}^{\infty} c_a(\nu)\exp(2\pi i H'\nu/k)\exp\{-(\pi/Kkw)(2\nu-(1/6)) + \pi w(2n-(A/6p))\}d\phi, \\ &\qquad 0 \leq h < k \leq N, \qquad \theta' = \theta'_{h,k}, \qquad \theta'' = \theta''_{h,k}. \end{aligned} \tag{7.6}$$

As before, we split this sum into two parts $Q'(n)$ and $R'(n)$. Here $Q'(n)$ contains only the value $\nu = 0$. We find, as before,

$$R'(n) = O(\exp(c\pi N^{-2})n^{1/3}N^{-1/3+\epsilon}) \tag{7.7}$$

and

$$Q'(n) = \pi\sum_{k=1}^{N} \csc(\pi\alpha/p)B_k(n)L'_k(n) + O(\exp(c\pi N^{-2})n^{1/3}N^{-1/3+\epsilon}). \tag{7.8}$$

where

$$B_k(n) = B(n, 0;\ k;\ 0, k;\ a) \tag{7.81}$$

and

$$\begin{aligned} L'_k(n) &= \frac{1}{k(12pn-A)^{1/2}} \quad I_1\left\{\frac{\pi(12pn-A)^{1/2}}{3pk}\right\} \quad \text{for} \quad n > A/12p \\ &= \frac{1}{k(A-12pn)^{1/2}} \quad J_1\left\{\frac{\pi(A-12pn)^{1/2}}{3pk}\right\} \quad \text{for} \quad n < A/12p. \end{aligned} \tag{7.82}$$

Upon combining (7.3), (7.5), (7.7), and (7.8), keeping n fixed and letting $N \to \infty$, we obtain our desired result.

THEOREM 4. *The number, $p_a(n)$, of partitions of a positive integer n into positive summands of the form $pl \pm a$ $(a = 1, 2, \cdots, (p-1)/2)$ is given by the convergent series*

$$\begin{aligned} p_a(n) &= 2\pi \sum_{\substack{k>0 \\ p|k}} \sum_{b=1}^{[p(3-\sqrt{3})/6]} \sum_{\rho=0}^{[B/12pb]} A_{k,b}(n)L_k(n) \\ &+ \pi \sum_{\substack{k>0 \\ p\nmid k}} B_k(n)\,|\csc \pi ak/p\,|\, L'_k(n) \end{aligned} \tag{7.9}$$

where $L_k(n)$ *and* $L'_k(n)$ *are given by* (7.52) *and* (7.82), $A_{k,b}(n)$ *by* (7.51), *and* $B_k(n)$ *by* (7.81).

8. To determine the asymptotic ratios of the $p_a(n)$, we have to investigate the dominant terms in each of the series of (7.9). Inspection of these series clearly shows that the leading term in each series dominates all others. Thus, we must have

$$(8.1)\qquad \begin{array}{l} k = p, \quad b = 1, \quad \rho = 0 \quad \text{if} \quad p \mid k \\ k = 1 \quad \text{if} \quad p \nmid k. \end{array}$$

In brief, the dominant terms are

$$(8.2)\qquad \frac{2\pi}{(12pn - A)^{\frac{1}{2}}} A_{p,1}(n)(1/p)B_1^{\frac{1}{2}} \quad I_1\left\{\frac{\pi(12pn - A)^{\frac{1}{2}}B_1^{\frac{1}{2}}}{3p^2}\right\}$$

in the series for $p \mid k$ and

$$(8.3)\qquad \frac{\pi}{(12pn - A)^{\frac{1}{2}}} B_1(n) \,|\csc \pi a/p\,| \quad I_1\left\{\frac{\pi(12pn - A)^{\frac{1}{2}}}{3p}\right\}$$

in the series for $p \nmid k$.

From (2.81) and (8.1), we find

$$B_1 = p^2 - 6p + 6.$$

Hence,

$$(8.4)\qquad B_1/p^2 = \frac{p^2 - 6p + 6}{p^2} = 1 - 6/p + 6/p^2 < 1.$$

Therefore

$$(8.5)\qquad \frac{\pi(12pn - A)^{\frac{1}{2}}B_1^{\frac{1}{2}}}{3p^2} < \frac{\pi(12pn - A)^{\frac{1}{2}}}{3p}.$$

From the theory of the Bessel functions, we obtain the following estimates [10]

$$\begin{array}{ll} I_1(z) = O(z), & |z| < 1 \\ I_1(z) \sim \exp z \cdot (2\pi z)^{-\frac{1}{2}}, & |z| > 1. \end{array}$$

By virtue of (8.2), (8.3), (8.4), and (8.5) we note that the term originating from the series for the case $p \nmid k$ dominates that originating from the series for the case $p \mid k$. Hence we obtain

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[10] See [8], page 203.

$$p_1(n) : p_2(n) : \cdots : p_{(p-1)/2}(n)$$

$$\sim \frac{|\csc \pi/p|}{(12pn - A_1)^{1/2}} I_1 \left\{ \frac{\pi(12pn - A_1)^{1/2}}{3p} \right\} :$$

$$\frac{|\csc 2\pi/p|}{(12pn - A_2)^{1/2}} I_1 \left\{ \frac{\pi(12pn - A_2)^{1/2}}{3p} \right\} : \cdots :$$

$$\frac{|\csc (p-1)\pi/2p|}{(12pn - A_{(p-1)/2})^{1/2}} I_1 \left\{ \frac{\pi(12pn - A_{(p-1)/2})^{1/2}}{3p} \right\} .$$

Applying (8.6), we finally obtain

$$p_1(n) : p_2(n) : \cdots : p_{(p-1)/2}(n)$$
$$\sim \csc \pi/p : \csc 2\pi/p : \cdots : \csc (p-1)\pi/2p.$$

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ON THE LACK OF AN EUCLIDEAN ALGORITHM IN $R(\sqrt{61})$.*

By L. K. Hua and W. T. Shih.

It has been proved by A. Brauer [1] that the fields $R(\sqrt{p})$, p a prime $\equiv 13 \pmod{24}$, are non-Euclidean, except possibly for $p = 13$, 37, 61 and 109. In the cases $p = 13$ and 37 the fields $R(\sqrt{p})$ are Euclidean. Thus it is an open question whether the fields $R(\sqrt{61})$ and $R(\sqrt{109})$ are Euclidean or not,

Theorem 1. *The quadratic field $R(\sqrt{61})$ is non-Euclidean*

It is known that Theorem 1 is a consequence of

Theorem 2. *For any integers x, y we have always*

$$|(x + (1/2)y - 1/78)^2 - (61/4)(y - 17/39)^2| > 1.$$

Let

$$u = 78x + 39y - 1, \qquad v = 39y - 17,$$

If Theorem 2 is false, then we have integers u, v and k such that

$$u^2 - 61v^2 = k, \quad \text{where} \quad |k| \leq 78^2 = 6084. \tag{1}$$

Since

$$u \equiv -1, \qquad v \equiv -17 \pmod{39}, \tag{2}$$

we have

$$k \equiv 1^2 + 17^3 \equiv 0 \pmod{39};$$

furthermore since

$$k \equiv (y+1)^2 - (y+1)^2 = 0 \pmod 4,$$

we have that k is a multiple of 156 $(= 4 \times 39)$. Moreover, evidently $\left(\dfrac{k}{61}\right) = 1$, since k is not a multiple of 61 (in fact $61 \times 156 > 6084$). By these two properties we may list all the possible values of l, where $k = 156l$ and $|l| \leq 39$:

0, ± 1, ± 3, ± 4, ± 5, ± 9, ± 12, ± 13, ± 14, ± 15, ± 16, ± 19, ± 20, ± 22, ± 25, ± 27, ± 34, ± 36, ± 39.

* Received April 25, 1944.

[1] Brauer, *American Journal of Mathematics*, vol. 62 (1940), pp. 697-716.

For $l = 0, \pm 14, \pm 22, \pm 34$, (1) is insoluble. In fact, e. g., if

$$u^2 - 61v^2 = \pm 14 \times 156,$$

then

$$u^2 \equiv 61v^2 \pmod 7,$$

and $\left(\frac{61}{7}\right) = -1$, which is impossible.

We now give the solutions of (1) for these possible values of k.

Lemma 1. *The numbers of primary*[2] *solutions of* (1) *are given by the following table:*

l	± 1	± 3	± 4	± 5	± 9	± 12	± 13	± 15	± 16	± 19	± 20	± 25	± 27	± 36	$\pm$[illegible]
no. of primary solutions of (1)	4	6	4	8	8	6	6	12	4	8	8	12	10	8	[illegible]

The proof of the lemma is a straightforward calculation of

$$\sum_{n|k} \left(\frac{61}{n}\right),$$

which is the number of primary representations of k.

Lemma 2. *If* $u^2 - 61v^2 = k$ *has no primary solution satisfying*

(2′) $$u = \pm 1, \qquad v = \pm 17 \ (\mathit{mod}\ 39),$$

then $u^2 - 61v^2 = \pm k$ *has no solution satisfying* (2′).

Proof. Since $\eta = (39 + 5\sqrt{61})/2$ is the fundamental solution of $u^2 - 61v^2 = -4$ (in fact $39^2 - 615^2 = -4$), the solutions of $u^2 - 61v^2 = \pm k$ are given by

$$\frac{u + v\sqrt{61}}{2} = \pm \left(\frac{39 + 5\sqrt{61}}{2}\right)^{\lambda} \left(\frac{u_0 + v_0\sqrt{61}}{2}\right), \qquad \lambda \lesseqgtr 0,$$

where u_0, v_0 is a solution of (1).

For $\lambda = 1$, we have

(3) $$u = \pm \tfrac{1}{2}(39u_0 + 5 \times 61v_0), \qquad v = \pm \tfrac{1}{2}(39v_0 + 5u_0),$$

i. e.

$$u \equiv \pm 16v_0, \qquad v \equiv \pm 17u_0, \qquad \pmod{39},$$

i. e.

$$u_0 \equiv \pm 16v, \qquad v_0 \equiv \pm 17u, \qquad \pmod{39}$$

(since $16 \times 17 \equiv -1 \pmod{39}$). If $u \equiv \pm 1$, $v \equiv \pm 17$, then we have $u_0 \equiv \pm 1$, $v_0 \equiv \pm 17 \pmod{39}$.

[2] Landau, *Vorlesungen über Zahlentheorie*, Bd. 1, S. 140.

The proof is easily completed by repeated application of the same argument.

Thus in order to prove Theorem 2 it remains only to establish that there is no primary solution of (1) with (2) and $0 < l \leq 39$.

LEMMA 3. *All the primary solutions of* (1) *are given by the following table:*

l	u, v
1	20, 2; 41, 5; 1484, 190; 3335, 427.
3	23, 1; 38, 4; 282, 36; 633, 81; 5147, 659; 11567, 1481.
4	40, 4; 82, 10; 2968, 380; 6670, 854.
5	29, 1; 32, 2; 151, 19; 337, 43; 883, 113; 1984, 254; 12473, 1597; 16136, 2066.
9	60, 6; 123, 15; 245, 31; 548, 70; 977, 125; 2195, 281; 4452, 570; 10005, 1281.
12	46, 2; 76, 8; 564, 72; 1266, 162; 10294, 1318; 23134, 2962.
13	97, 11; 208, 26; 463, 59; 1672, 214; 3757, 481; 8443, 1081.
15	49, 1; 73, 7; 134, 16; 171, 21; 293, 37; 378, 48; 2367, 303; 3062, 392; 5319, 681; 6881, 881; 13957, 1787; 31366, 4016.
16	80, 8; 164, 20; 5936, 760; 13340, 1708.
19	55, 1; 67, 5; 433, 55; 970, 124; 1165, 149; 2617, 335; 21283, 2725; 35935, 4601.
20	58, 2; 64, 4; 302, 38; 674, 86; 1766, 226; 3968, 508; 24946, 3194; 32272, 4132.
25	83, 7; 100, 10; 161, 19; 205, 25; 527, 67; 1181, 151; 1259, 161; 2828, 362; 7420, 950; 9599, 1229; 16675, 2135; 21572, 2762.
27	69, 3; 114, 12; 191, 23; 419, 53; 846, 108; 1899, 243; 3851, 493; 8654, 1108; 15441, 1977; 34701, 4443.
36	120, 12; 246, 30; 490, 62; 1096, 140; 1954, 250; 4390, 562; 8904, 1140; 20010, 2562.
39	78, 0; 105, 9; 322, 40; 715, 91; 1447, 185; 1603, 205; 3250, 416; 7303, 935; 26430, 3384.

By Lemma 1, the table contains all primary solutions. The verification is straightforward.

Moreover no solution contained in the table satisfies (2′). Thus Theorem 2 is established.

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THE PROBLEM OF MILLOUX FOR FUNCTIONS ANALYTIC THROUGHOUT THE INTERIOR OF THE UNIT CIRCLE.*

By Maurice H. Heins.

1. **Introduction.** Let E denote a point set of the z-plane lying in, and closed relative to, $|z| < 1$, having the property that for all R satisfying: $0 \leqq R < 1$ the intersection of E with $|z| = R$ is not void. A function $f(z)$ will be said *to belong to the class* $\mathfrak{F}_m$, if $f(z)$ is defined, analytic, and of modulus less than one for $|z| < 1$ and if there exists an E such that $z \in E$ implies $|f(z)| \leqq m$, where m is a given positive number less than one. Let $M(f; r)$ have its usual significance as $\max_{|z|=r} |f(z)|$. *It is required to determine* $\underset{f \in \mathfrak{F}_m}{\text{l.u.b.}} M(f; r)$ $(0 < r < 1)$ *and the associated extremal functions.*

This problem is solved completely in the present paper; it will be shown that the extremal function suitably normalized is unique and independent of r. The normalized extremal function will be determined and expressed with the aid of the Jacobi theta functions. Two methods of obtaining the extremal function are indicated. One involves the theory of Blaschke products; the other gives an algorithm for determining the extremal function by the repeated composition of $(1, 2)$ directly conformal maps of the interior of the unit circle onto itself.

The methods developed are applicable to other problems. Analogues of Tchebycheff polynomials in the hyperbolic plane are defined and determined.

The problem of the present paper is closely related to a series of investigations initiated by Milloux [9], who showed that, if $f(z)$ is analytic and of modulus less than or equal to one for $|z| \leqq 1$ and if there exists a Jordan arc J lying in $|z| < 1$, except for one endpoint on $|z| = 1$, and joining $z = 0$ to the circumference $|z| = 1$ with $|f(z)| \leqq m$ (< 1) for $z \in J$, then there exists a positive constant k, *independent of* m *and* J, such that

$$(1.1) \qquad |f(z)| \leqq m^{k[1-|z|]} \qquad (|z| < 1).$$

Lower and upper estimates for the best value of k have been given by Landau [8]. Subsequently, E. Schmidt [14] considered the problem of Milloux under the weaker hypotheses which assumed: (a) $f(z)$ to be analytic and of modulus less than one for $z \in \mathfrak{G}$ where $\mathfrak{G} \equiv [|z| < 1] \cdot CJ$, J having the same connota-

* Received January 11, 1944.

tion as above; (b) $\overline{\lim}_{z(\epsilon \mathfrak{G}) \to J} |f(z)| \leqq m$, and showed the existence of a positive k, again independent of m and J, such that (1.1) held for the $f(z)$ considered and $z \epsilon \mathfrak{G}$. In fact, he showed the best value of k for the problem he treated to be $1/\pi$ and indicated that this was also the best value of k when $f(z)$ is actually defined and analytic throughout $|z| < 1$. The problem was later studied under broader hypotheses by R. Nevanlinna [12], Beurling [2], Fenchel [4], and Milloux [10], but always with the view to obtaining inequalities having universal constants independent of m and the set E where $\overline{\lim}_{z \to E} |f(z)| \leqq m$.

It should be emphasized that the present investigation is concerned with functions analytic throughout the interior of the unit circle and deals with m fixed.

2. Preliminary reductions. Let $g(z)$ be defined, analytic and of modulus less than one for $|z| < 1$. Recall that $g(z)$ can be represented [1] by

$$(2.1) \qquad g(z) = \pi(z) \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta) + i\lambda \right\},$$

where $\mu(\theta)$ is defined and monotone non-increasing for $0 \leqq \theta \leqq 2\pi$, λ is a real constant and $\pi(z)$ is defined to be (a) identically equal to one if $g(z)$ has no zeros for $|z| < 1$, (b) the finite product

$$(2.2) \qquad \prod_{k=1}^{n} \frac{\bar{\alpha}_k}{|\alpha_k|} \cdot \frac{\alpha_k - z}{1 - \bar{\alpha}_k z}$$

if $g(z)$ has a finite number of zeros $\alpha_1, \cdots, \alpha_n$, multiplicities being counted as usual (if any $\alpha_k = 0$, then the corresponding factor is to be replaced simply by z), (c) the Blaschke product [2]

$$(2.3) \qquad \prod_{k=1}^{\infty} \frac{\bar{\alpha}_k}{|\alpha_k|} \cdot \frac{\alpha_k - z}{1 - \bar{\alpha}_k z}$$

if $g(z)$ has an infinite set of zeros $\alpha_1, \alpha_2, \cdots$, the same conventions prevailing here and throughout the discussion as in the case of the finite product (2.2).

Suppose now that $f_0(z) \epsilon \mathfrak{F}_m$ is extremal and that E_0 is a corresponding E-set.[3] By means of a trivial normalization it may be assumed that $f_0(r)$

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[1] This is essentially the representation due to Herglotz [6]. Cf. Evans [3] and R. Nevanlinna [11] for accounts of this representation. The use of the Poisson-Stieltjes representation is not necessary in this connection but is suggestive from a physical point of view.

[2] An exposition of the theory of Blaschke products as well as of the products (2.2) is to be found in Julia [7].

[3] The proof of the existence of an extremal function and an associated E-set is readily established and will be omitted.

$= M(f_0; r)$. The object of the present section is to show that, subject to this normalization, the set $[-1 < x \leqq 0]$ may be regarded as an associated E-set of $f_0(z)$. Define $\pi^*(z)$ to be identically equal to one if $f_0(z)$ is free from zeros;

$$\pi^*(z) \equiv \prod_{k=1}^{n} \frac{|z_k| + z}{1 + |z_k| z} \tag{2.3}$$

if $f_0(z)$ has a finite number of zeros $z_1, \cdots, z_n$;

$$\pi^*(z) \equiv \prod_{k=1}^{\infty} \frac{|z_k| + z}{1 + |z_k| z} \tag{2.4}$$

if $f_0(z)$ has an infinite set of zeros $z_1, z_2, \cdots$. Now let $h(z)$ denote

$$h(z) \equiv \pi^*(z) \exp\left\{ \frac{1-z}{1+z} \cdot \frac{1}{2\pi} \int_0^{2\pi} d\mu(\theta) \right\} \tag{2.5}$$

where $\mu(\theta)$ has the connotation of (2.1) for $f_0(z)$. If $z \in E_0$, then

$$|h(-|z|)| \leqq |f_0(z)|, \tag{2.6}$$

and

$$h(r) \geqq f_0(r), \tag{2.7}$$

equality prevailing in (2.7) if and only if $h(z) \equiv f_0(z)$. These facts may be readily ascertained by the use of standard inequalities for expressions of the form

$$\frac{z-\alpha}{1-\bar{\alpha}z} \quad (|\alpha| < 1)$$

and for the Poisson-Stieltjes integral (inequalities of the Harnack type). Inequalities (2.6) and (2.7) imply, respectively, that $h(z) \in \mathfrak{F}_m$ with $[-1 < x \leqq 0]$ as an associated E-set and that $h(z)$ is extremal. Since the strong inequality prevails in (2.7) if $h(z) \not\equiv f_0(z)$, it follows that, if $f_0(z)$ is extremal, then $h(z) \equiv f_0(z)$, and hence

THEOREM 2.1. *If $f_0(z)$ is extremal and is so normalized that $M(f_0; r) = f_0(r)$, then $[-1 < x \leqq 0]$ is an associated E-set.*

3. The qualitative characterization of the extremal function. If $f_0(z)$ is extremal and normalized as in **2**, then $f_0(z)$ is also an extremal function for the following problem.

PROBLEM. *Let $f(z)$ be analytic and of modulus less than one for $|z| < 1$. Further let it be required that $|f(z)| \leqq m < 1$, m being a given positive*

number less than one, for z real and non-positive $(-1 < z \leqq 0)$. *Finally let r be real and lie strictly between zero and one. It is required to determine*

$$\underset{\{f\}}{\text{l. u. b.}}\, |f(r)|$$

and the associated extremal functions.

This problem has been studied qualitatively by the author,[4] the pertinent results being summarized in

THEOREM 3.1. *If $f(z)$ is so normalized that $f(r) > 0$, then there exists an extremal function, $\phi(z; m, r)$, and it is unique. It has the following properties:*

(a) *$\phi(z; m, r)$ is a Blaschke product;*

(b) *$\phi(z; m, r)$ has an infinite number of zeros, all simple and negative;*

(c) *$\phi(z; m, r)$ is real for z real;*

(d) *on the real axis between successive zeros of $\phi(z; m, r)$ there exists a unique point where $\phi^2 = m^2$;*

(e) *$\phi(0; m, r) = m$, $\phi'(0; m, r) > 0$;*

(f) *ϕ' has simple zeros at the points whose existence is affirmed in* (d) *and has no other zeros;*

(g) *the zeros of $\phi^2 - m^2$ are $z = 0$ and the points defined in* (d), *and there are no others.*

Let it be agreed that $G[w(z)]$ will denote the Riemannian image of $|z| < 1$, with respect to $w(z)$, where $w(z)$ is understood to be analytic for $|z| < 1$.

From Theorem 3.1 the structure of $G[\phi]$ can be readily inferred. It follows from (f) that $G[\phi]$ has branch-points only over $-m$ and m. Consider now the Riemann domain S lying over $|w| < 1$ which is defined as follows. Let S_1 denote the open simple disk $|w| < 1$ slit along the real axis from -1 to $-m$, and let S_k $(k \geqq 2)$ denote copies of the open disk $|w| < 1$ slit along the real axis from -1 to $-m$ and from m to $+1$. The Riemann domain S is constructed by joining the upper bank of the slit of S_1 to the lower bank of the left-hand slit of S_2 and the lower bank of the slit of S_1 to the upper bank of the left-hand slit of S_2, then by joining in the same fashion the banks of the right-hand slits of S_2 and S_3, then the banks of the left-hand slits of S_3 and S_4, and so forth. The Riemann domain, S, obtained in this

[4] Cf. M. H. Heins [5].

fashion is simply-connected and bounded. Hence by the fundamental mapping theorem of conformal mapping there exists a function $w = \Phi(z; m)$ which is analytic and of modulus less than one for $|z| < 1$, which satisfies the normalizing conditions:

$$\Phi(0; m) = m, \qquad \Phi'(0; m) > 0,$$

and maps $|z| < 1$ onto the Riemann domain S.

Consider now the relation

$$\Phi[\psi(z); m] = \phi(z; m, r) \tag{3.1}$$

where $\psi(z)$ is normalized to vanish at $z = 0$. It is readily inferred from the structure of $S \equiv G[\Phi]$ that there exists a function, $\psi(z)$, which is analytic, single-valued, and of modulus less than one for $|z| < 1$, which vanishes at $z = 0$, and satisfies (3.1) identically for all z in $|z| < 1$. Next, from the structure of S and the normalization requirements imposed on Φ, it may be inferred that Φ is real and monotone increasing for z real and positive, and that Φ is real and of modulus not exceeding m for z real and negative. Finally, $\psi(z)$ is real and positive (in fact monotone increasing) for z real and positive. Hence $\Phi(z; m)$ is a competitor in the extremal problem proposed at the beginning of this section. Also $\psi(r) \leqq r$ $(0 < r < 1)$ where equality prevails if and only if $\psi(z) \equiv z$. From (3.1) and the enumerated properties of Φ and ψ it follows that

$$\Phi(r; m) \leqq \phi(r; m, r) = \Phi(\psi(r); m) \leqq \Phi(r; m). \tag{3.2}$$

Hence $\psi(r) = r$, $\Phi(z; m) \equiv \phi(z; m, r)$, and $S \equiv G[\phi]$. One important consequence of this result is

THEOREM 3.2. *The extremal function* $\phi(z; m, r)$ *is independent of* r.

For the remainder of the present paper the extremal function will be denoted simply by $\phi(z; m)$.

The information obtained about the structure of $G[\phi]$ will be very valuable for the explicit determination of $\phi(z; m)$. An immediate application of the present results is a recurrence relation for $\phi(z; m)$ which may be found as follows. Consider the function $\phi^*(z)$ defined by

$$[\phi^*(z)]^2 \equiv \frac{\phi(z; m) + m}{1 + m\phi(z; m)}; \qquad \phi^*(0) > 0. \tag{3.3}$$

The function $\phi^*(z)$ may be continued to be analytic throughout $|z| < 1$, since at the points where $\phi(z; m) = -m$ it is true that $\phi'(z; m)$ has a simple zero. The Riemann domain, $G[\phi^*]$, has the same ramification proper-

ties as $G[\phi]$ except that the branch points are now over $\sqrt{2m/(1+m^2)}$ and $-\sqrt{2m/(1+m^2)}$. The conditions (3.3) imposed on $\phi^*(z)$ imply that the derivative of $\phi^*(z)$ at $z=0$ is positive. Hence $\phi^*(z) \equiv \phi(z; \sqrt{2m/(1+m^2)})$ and (3.3) yields

$$[\phi(z; \sqrt{2m/(1+m^2)})]^2 \equiv \frac{\phi(z;m)+m}{1+m\phi(z;m)}. \tag{3.4}$$

4. The determination of $\varphi(z;m)$ by a process of successive compositions of rational functions. In this section an algorithm will be developed for determining $\phi(z;m)$ as the limit of a sequence of rational functions. One noteworthy feature of this algorithm is that each member of the sequence provides a majorant for $\phi(r;m)$ $(0<r<1)$ and hence furnishes incidentally an upper bound for $M(f;r)$ where $f \in \mathfrak{F}_m$.

Instead of considering $\phi(z;m)$ directly, it will be convenient to study

$$\omega(z) \equiv \frac{\phi(z;m)-m}{1-m\phi(z;m)}. \tag{4.1}$$

The structure of the Riemann domain, $G[\omega]$, is the same as that of $G[\phi]$ save that the ramification points of $G[\omega]$ lie over 0 and $-2m/(1+m^2)$. Let μ denote $2m/(1+m^2)$. Define $\omega_1(z)$ by the relation

$$\left[\frac{\omega_1(z)+k_1}{1+k_1\omega_1(z)}\right]^2 = \frac{\omega(z)+\mu}{1+\mu\omega(z)}, \tag{4.2}$$

and the requirements: $\omega_1(0)=0$, $\omega'_1(0)>0$, k_1 being real and so determined that these conditions are fulfilled.[5] It is clear that, subject to these conditions, $k_1{}^2=\mu$ and on differentiating (4.2) with respect to z that

$$2\,\frac{\omega_1(0)+k_1}{1+k_1\omega_1(0)}\cdot\frac{(1-k_1{}^2)\omega'_1(0)}{[1+k_1\omega_1(0)]^2} = \frac{(1-\mu^2)\omega'(0)}{[1+\mu\omega(0)]^2}$$

or

$$2k_1(1-k_1{}^2)\omega'_1(0) = (1-\mu^2)\omega'(0).$$

Hence $k_1>0$ and is equal to $\sqrt{\mu}$. With k_1 so specified, $\omega_1(z)$ is uniquely determined as a function analytic and of modulus less than unity for $|z|<1$. A study of the Riemann domain, $G[\omega_1]$, yields the result that $G[\omega_1]$ has the same structure as $G[\omega]$ but with its ramification points lying over 0 and $-(2\sqrt{\mu}/(1+\mu))$; $(2\sqrt{\mu}/(1+\mu)$ will be denoted by $\mu_1)$. From (4.2) it follows that $\omega_1(z)$ is real, monotone increasing and positive for z real and positive since $\omega(z)$ itself has this property. But then from (4.1) and (4.2)

[5] It is to be observed that this is essentially the device used by Koebe for establishing the fundamental mapping theorem.

$$(4.3)\quad \phi(z;m) \equiv \frac{\omega(z)+m}{1+m\omega(z)} = \frac{\left(\dfrac{\omega_1+\sqrt{\mu}}{1+\sqrt{\mu}\,\omega_1}\right)^2 + \left(\dfrac{m-\mu}{1-m\mu}\right)}{1+\left(\dfrac{m-\mu}{1-m\mu}\right)\left(\dfrac{\omega_1+\sqrt{\mu}}{1+\sqrt{\mu}\,\omega_1}\right)^2}$$

$$= \frac{\left(\dfrac{\omega_1+\sqrt{\mu}}{1+\sqrt{\mu}\,\omega_1}\right)^2 - m}{1-m\left(\dfrac{\omega_1+\sqrt{\mu}}{1+\sqrt{\mu}\,\omega_1}\right)^2}$$

and since, by Schwarz's lemma, $\omega_1(r) < r$ for r real and positive, a rational majorant for $\phi(r;m)$ is given by replacing $\omega_1(r)$ by r. Hence

$$(4.4)\qquad \phi(r;m) < \frac{\left(\dfrac{r+\sqrt{\mu}}{1+\sqrt{\mu}\,r}\right)^2 - m}{1-m\left(\dfrac{r+\sqrt{\mu}}{1+\sqrt{\mu}\,r}\right)^2}.$$

The process applied to $\omega(z)$ to determine $\omega_1(z)$ may now be applied to $\omega_1(z)$ with μ_1, $\sqrt{\mu_1}$, $\omega_1(z)$, $\omega_2(z)$ replacing μ, k_1, $\omega(z)$, $\omega_1(z)$, respectively, in (4.2) and the conditions $\omega_2(0)=0$, $\omega'_2(0)>0$ replacing the analogous conditions on $\omega_1(z)$. This process can be continued step-by-step indefinitely yielding a sequence of functions $\{\omega_k(z)\}$ and a sequence of positive constants $\{\mu_k\}$ $(k=0,1,\cdots)$ where

$$(4.5)\qquad \mu_0=\mu,\qquad \mu_{k+1}=\frac{2\sqrt{\mu_k}}{1+\mu_k}\qquad (k=0,1,\cdots);$$

$$(4.6)\qquad \omega_0(z)\equiv\omega(z),\ \left[\frac{\omega_{k+1}+\sqrt{\mu_k}}{1+\sqrt{\mu_k}\,\omega_{k+1}}\right]^2 \equiv \frac{\omega_k+\mu_k}{1+\mu_k\omega_k}\qquad (k=0,1,\cdots);$$

and

$$(4.7)\qquad \omega_k(0)=0,\qquad \omega'_k(0)>0\qquad (k=0,1,\cdots).$$

Let $q_k(Z)$ be defined by

$$(4.8)\qquad q_k(Z) \equiv \frac{\left(\dfrac{Z+\sqrt{\mu_k}}{1+\sqrt{\mu_k}\,Z}\right)^2 - \mu_k}{1-\mu_k\left(\dfrac{Z+\sqrt{\mu_k}}{1+\sqrt{\mu_k}\,Z}\right)^2}\qquad (k=0,1,\cdots)$$

and let $Q_k(Z)$ be defined by the relations

$$(4.9)\qquad Q_0(Z)\equiv q_0(Z);\qquad Q_k(Z)\equiv Q_{k-1}[q_k(Z)]\qquad (k=1,2,\cdots).$$

It is an immediate consequence of these definitions that

$$\phi(z;m) \equiv \frac{Q_k[\omega_{k+1}(z)] + m}{1 + mQ_k[\omega_{k+1}(z)]} \qquad (k = 0, 1, \cdots). \tag{4.10}$$

An argument of the type used to deduce (4.4) yields the following sequence of majorants for $\phi(r:m)$ (r real and between 0 and 1):

$$\phi(r;m) < \frac{Q_k(r) + m}{1 + mQ_k(r)} \qquad (k = 0, 1, \cdots). \tag{4.11}$$

It is natural to conjecture that

$$\phi(z;m) = \lim_{k\to\infty} \frac{Q_k(z) + m}{1 + mQ_k(z)} \quad \text{for} \quad |z| < 1.$$

This may be proved from properties of $Q_k(z)$ resulting solely from its definition. From (4.9) it follows that the degree of $Q_k(z)$ is 2^{k+1}. Second, consider the image of the interval $[-\mu_{k+1} \leqq x \leqq 0]$ with respect to $Q_k(x)$. For $k = 0$, the image of $[-\mu_1 \leqq x \leqq 0]$ with respect to $Q_0(x)$ is $[-\mu \leqq u \leqq 0]$. This may be seen by observing that the minimum value attained by $Q_0(x)$ for real x is $-\mu$ and that this minimum is attained at $x = -\sqrt{\mu}$ which evidently satisfies the inequality

$$-\mu_1 = -\frac{2\sqrt{\mu}}{1+\mu} < -\sqrt{\mu} < 0.$$

To obtain the image of $[-\mu_2 \leqq x \leqq 0]$ with respect to $Q_1(x)$, recall that $Q_1(z) \equiv Q_0[q_1(z)]$ and observe that the image of $[-\mu_2 \leqq x \leqq 0]$ with respect to $q_1(x)$ is $[-\mu_1 \leqq X \leqq 0]$; this last remark follows by the argument used on $Q_0(x)$ and $[-\mu_1 \leqq x \leqq 0]$. Hence the image of $[-\mu_2 \leqq x \leqq 0]$ with respect to $Q_1(x)$ is $[-\mu \leqq u \leqq 0]$. By observing that the image of $[-\mu_{k+1} \leqq x \leqq 0]$ with respect to $q_k(x)$ is $[-\mu_k \leqq X \leqq 0]$ and recalling the defining relations (4.9) the proof of the original assertion is established.

To complete the proof of the conjecture, note that $\lim_{k\to\infty} \mu_k = 1$ (this fact follows from the elementary observation that $\{\mu_k\}$ is a positive monotone increasing sequence with limit $\mu^* \leqq 1$, and that $\mu^* = 2\sqrt{\mu^*}/(1+\mu^*)$). Let $\{Q_{k(n)}(z)\}$ denote any subsequence of $\{Q_k(z)\}$ which converges for $|z| < 1$ and let $Q^*(z)$ denote the corresponding limit function. Since for each k $(k = 0, 1, 2, \cdots)$ the image of $[-\mu_{k+1} \leqq x \leqq 0]$ with respect to $Q_k(z)$ is $[-\mu \leqq u \leqq 0]$ and since $\lim_{k\to\infty} \mu_k = 1$, the limit function $Q^*(z)$ must have the property that the image of $[-1 < x \leqq 0]$ with respect to $Q^*(z)$ must belong to the interval $[-\mu \leqq u \leqq 0]$. Hence the function

$$\frac{Q^*(z) + m}{1 + mQ^*(z)}$$

must be a competitor in the extremal problem of **3**. But then it is clear that

$$\frac{Q^*(r) + m}{1 + mQ^*(r)} \leqq \phi(r; m),$$

and on the other hand (4.11) implies

$$\phi(r; m) \leqq \frac{Q^*(r) + m}{1 + mQ^*(r)}.$$

In other words,

$$\phi(z; m) \equiv \frac{Q^*(z) + m}{1 + mQ^*(z)}.$$

Now the sequence $\{Q_k(z)\}$ does converge. If it did not, there would exist two subsequences with distinct limit functions. However the argument just applied to $Q^*(z)$ shows that any limit function must coincide with

$$\frac{\phi(z; m) - m}{1 - m\phi(z; m)}.$$

Hence $\lim\limits_{k\to\infty} Q_k(z)$ exists for $|z| < 1$ and stands in the indicated relation to $\phi(z; m)$.

THEOREM 4.1. *If $f(z) \in \mathfrak{F}_m$, then $M(f; r) < \dfrac{Q_k(r) + m}{1 + mQ_k(r)}$ $(k = 0, 1, 2, \cdots)$. Further, the sequence $\{Q_k(z)\}$ converges continuously for $|z| < 1$ and*

$$\phi(z; m) \equiv \frac{\lim\limits_{k\to\infty} Q_k(z) + m}{1 + m[\lim\limits_{k\to\infty} Q_k(z)]}.$$

5. The representation of $\varphi(z; m)$ as a Blaschke product. Let $B(z; m)$ denote $\phi(-z^2; m)$. Clearly $B(z; m)$ is a Blaschke product which is even and has all its zeros simple and real. Another consequence of the definition of $B(z; m)$ is that $|B(z; m)|$ has m as its maximum between successive zeros of $B(z; m)$ (for z real). The Riemann domain, $G[B(z; m)]$, is completely ramified (the covering is always twofold locally) over m and $-m$. Hence if α denotes the smallest positive zero of $B(z; m)$, and if T denotes the linear fractional transformation of $|z| < 1$ onto itself with fixed points -1 and $+1$, which carries $-\alpha$ into α, then

$$B(T; m) = -B(z; m). \tag{5.1}$$

The zeros of $B(z; m)$ are therefore $\{T^k\alpha\}$ $(k = 0, 1, 2, \cdots; -1, -2, \cdots)$.

For by (5.1) $T^k\alpha$ is a zero of $B(z;m)$ for every integer k. Nor can there be any other zero since such a zero would have a homologue with respect to some integral power of T strictly in the open interval $(-\alpha, \alpha)$ and this is contrary to the definition of α. It now remains to determine α and T in terms of m.

The transformation T may be expressed in the form

$$\frac{Tz-1}{Tz+1}=\lambda\,\frac{z-1}{z+1} \qquad (\lambda \text{ real}, 0<\lambda<1). \tag{5.2}$$

The condition, $T(-\alpha)=\alpha$, implies that $\lambda=\left(\frac{1-\alpha}{1+\alpha}\right)^2$. Hence T^k is given by

$$\frac{T^kz-1}{T^kz+1}=\left(\frac{1-\alpha}{1+\alpha}\right)^{2k}\cdot\frac{z-1}{z+1}, \tag{5.3}$$

or

$$T^kz=\frac{1-\left(\frac{1-\alpha}{1+\alpha}\right)^{2k}\left(\frac{1-z}{1+z}\right)}{1+\left(\frac{1-\alpha}{1+\alpha}\right)^{2k}\left(\frac{1-z}{1+z}\right)}. \tag{5.4}$$

Since $B(z;m)$ is an even Blaschke product which is positive for $z=0$, it must necessarily be given by

$$B(z;m)=\prod_{k=0}^{\infty}\frac{(T^k\alpha)^2-z^2}{1-(T^k\alpha)^2z^2}. \tag{5.5}$$

The value of α is determined by the condition $B(0;m)=m$. In other words, the relation

$$m=\prod_{k=0}^{\infty}(T^k\alpha)^2=\prod_{k=0}^{\infty}\left(\frac{1-\left(\frac{1-\alpha}{1+\alpha}\right)^{2k+1}}{1+\left(\frac{1-\alpha}{1+\alpha}\right)^{2k+1}}\right)^2 \tag{5.6}$$

must hold. It is readily verified that there exists a unique real α lying between 0 and 1 which satisfies (5.6) for a given positive m less than 1. This may be deduced by observing that the linear fractional transformation $\alpha \,\Big|\, \frac{1-\alpha}{1+\alpha}$ carries the interval $(0<\alpha<1)$ into itself and that the infinite product

$$P(\tau)\equiv\prod_{k=0}^{\infty}\left[\frac{1-\tau^{2k+1}}{1+\tau^{2k+1}}\right]^2$$

is analytic for $|\tau|<1$ and defines a monotone strictly decreasing map of the interval $(0<\tau<1)$ onto itself. As a matter of fact, the inversion of $P(\tau)$ is a classical problem in the theory of theta functions.

To recast the present problem in terms of the theory of theta functions, recall the following definitions and identities.[6] The functions $\theta_4(u)$ and $\theta_3(u)$ with parameter τ ($|\tau| < 1$) can be defined by

$$(5.7)\quad \begin{aligned} \theta_4(u) &\equiv 1 - 2\tau\cos 2\pi u + 2\tau^4\cos 4\pi u - 2\tau^9\cos 6\pi u + \cdots; \\ \theta_3(u) &\equiv 1 + 2\tau\cos 2\pi u + 2\tau^4\cos 4\pi u + 2\tau^9\cos 6\pi u + \cdots. \end{aligned}$$

For each τ satisfying $|\tau| < 1$ the functions $\theta_4(u)$ and $\theta_3(u)$ are entire in u. For the present purposes the following identities are significant:

$$(5.8)\quad \begin{aligned} \theta_4(u) &\equiv C\prod_{k=0}^{\infty}(1 - 2\tau^{2k+1}\cos 2\pi u + \tau^{4k+2}); \\ \theta_3(u) &\equiv C\prod_{k=0}^{\infty}(1 + 2\tau^{2k+1}\cos 2\pi u + \tau^{4k+2}), \end{aligned}$$

where C is independent of z and is obviously not zero—its exact value is of no concern for the problem at hand—and

$$(5.9)\quad \begin{aligned} \theta_4(0) &= C\prod_{k=0}^{\infty}(1 - \tau^{2k+1})^2; \\ \theta_3(0) &= C\prod_{k=0}^{\infty}(1 + \tau^{2k+1})^2. \end{aligned}$$

From (5.6) and (5.9)

$$(5.10)\quad m = \theta_4(0)/\theta_3(0)$$

where the parameter τ is equal to $(1-\alpha)/(1+\alpha)$. The formulas (5.8) will be employed in **6** to represent $\phi(z;m)$ in terms of the functions $\theta_4(u)$ and $\theta_3(u)$.

With α assumed determined as a function of m, $\phi(z;m)$ may be expressed immediately as a Blaschke product. The definition of $B(z;m)$ and the formula (5.5) imply

$$(5.11)\quad \phi(z;m) \equiv \prod_{k=0}^{\infty}\frac{(T^k\alpha)^2 + z}{1 + (T^k\alpha)^2 z}.$$

Note that the sequence $\{\phi_N(z)\}$, where $\phi_N(z) \equiv \prod_{k=0}^{N}\frac{(T^k\alpha)^2 + z}{1 + (T^k\alpha)^2 z}$, is monotone decreasing for any z real, positive, and less than 1. Hence $\phi_N(r)$ furnishes a rational majorant for $M(f;r)$ where $f \in \mathfrak{F}_m$ for $N = 0, 1, 2, \cdots$. By way of comparison with the methods developed in **4**, it is to be observed that the determination of the coefficients of ϕ_N involves the transcendental operation of computing α in terms of m, whereas the coefficients of $Q_k(z)$ may be found by essentially "algebraic" operations.

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[6] Following the notation used by Whittaker and Watson [16].

6. The representation of $\varphi(z; m)$ in terms of theta functions. With $\tau = (1-\alpha)/(1+\alpha)$ formulas (5.4) and (5.11) imply

$$\phi(z; m) = \prod_{k=0}^{\infty} \frac{\left(\dfrac{1-\tau^{2k+1}}{1+\tau^{2k+1}}\right)^2 + z}{1 + \left(\dfrac{1-\tau^{2k+1}}{1+\tau^{2k+1}}\right)^2 z} \tag{6.1}$$

or

$$\begin{aligned}\phi(z; m) &= \prod_{k=0}^{\infty} \frac{(1-\tau^{2k+1})^2 + z(1+\tau^{2k+1})^2}{(1+\tau^{2k+1})^2 + z(1-\tau^{2k+1})^2} \\ &= \prod_{k=0}^{\infty} \frac{(1+z) - 2\tau^{2k+1}(1-z) + \tau^{4k+2}(1+z)}{(1+z) + 2\tau^{2k+1}(1-z) + \tau^{4k+2}(1+z)} \\ &= \prod_{k=0}^{\infty} \frac{1 - 2\tau^{2k+1}\left(\dfrac{1-z}{1+z}\right) + \tau^{4k+2}}{1 + 2\tau^{2k+1}\left(\dfrac{1-z}{1+z}\right) + \tau^{4k+2}},\end{aligned}$$

and by (5.9)

$$\phi(z; m) = \theta_4(u)/\theta_3(u) \tag{6.2}$$

where τ is the parameter of both $\theta_4(u)$ and $\theta_3(u)$ and $\cos 2\pi u = (1-z)/(1+z)$.

7. A hyperbolic analogue of the Tchebycheff polynomials. It has been observed in **5** that $G[B(z; m)]$ is completely ramified over m and $-m$, having only branch points of order one over these points. Recall the structure of the Riemannian image of the finite z-plane with respect to $w = \cos z$, which is completely ramified over $+1$ and -1 with branch points of order one, and which is locally simple over every other point of the finite w-plane. On the strength of the analogy between the structure of $G[B(z; m)]$ and the Riemann surface for $\cos z$, it is natural to conjecture that in many respects $B(z; m)$ plays much the same role in the theory of functions which are analytic and bounded in the interior of the unit circle that $\cos z$ plays in the theory of entire functions. The similarity of the roles played by $B(z; m)$ and $\cos z$ will be brought out in two extremal problems to be considered. The first is

PROBLEM. *To determine among all* $(1, n)$ *directly conformal maps,* $\pi_n(z)$, *of* $|z| < 1$ *onto itself those for which*

$$\max_{-r \leqq x \leqq r} |\pi_n(x)|$$

is least and the corresponding écart. *Here n is a given positive whole number and r is a given positive number less than one.*

This problem is a hyperbolic analogue of the problem of determining the Tchebycheff polynomials from their extremal properties.

The first question that must be settled is that of existence. Let $\mu(n)$ denote $\underset{\{\pi_n(z)\}}{\text{g. l. b.}}\, [\max_{-r\leqq x\leqq r} |\pi_n(x)|]$. The class of functions consisting of all constants of modulus one and all directly conformal maps of $|z| < 1$ onto itself which are of degree not exceeding n is compact. Hence it may be shown by a familiar argument that there exists a member of this class, $\pi^*_k(z)$, of degree k $(0 \leqq k \leqq n)$ for which

$$\max_{-r\leqq x\leqq r} |\pi^*_k(x)| = \mu(n). \tag{7.1}$$

The degree k cannot be less than n. Otherwise the function $z^{n-k}\pi^*_k(z)$ would belong to the class $\{\pi_n(z)\}$ and would be such that

$$\max_{-r\leqq x\leqq r} |x^{n-k}\pi^*_k(x)| < \mu(n).$$

Hence there exists an extremal function in the class $\{\pi_n(z)\}$ for the present problem. A typical extremal function will henceforth be denoted by $\pi^*_n(z)$.

Qualitative properties of a typical extremal function may be readily determined. Of central importance among such properties for determining $\mu(n)$ and the extremal functions themselves are: 1° the location and the nature of the zeros, 2° the location of the points on $(-r \leqq x \leqq r)$ where $|\pi^*_n(z)|$ attains $\mu(n)$. Recall that a $(1, n)$ directly conformal map of $|z| < 1$ onto itself may be represented by

$$\pi_n(z) \equiv e^{i\theta} \prod_{k=1}^{n} \frac{z - \alpha_k}{1 - \bar{\alpha}_k z} \qquad (\theta \text{ real}; |\alpha_k| < 1, k = 1, 2, \cdots, n). \tag{7.2}$$

With the aid of this representation it can be shown that all the zeros of $\pi^*_n(z)$ are real. This will be demonstrated by using the inequality

$$\left| \frac{x - \Re\alpha}{1 - (\Re\alpha)x} \right| < \left| \frac{x - \alpha}{1 - \bar{\alpha}x} \right| \qquad (|\Re\alpha| < |\alpha| < 1) \tag{7.3}$$

where x is real and satisfies $-1 < x < 1$. The inequality may be established by direct analytical verification or else by noting that the non-Euclidean circle centered at x and tangent to the Euclidean straight line $\Re z = \Re\alpha$ does not contain α in its closed interior. On referring to the representation (7.2) one can see that by virtue of (7.3) if any zero of $\pi^*_n(z)$ is not real, $\mu(n)$ would be diminished if that zero were replaced by its real part.

Furthermore, every zero of $\pi^*_n(z)$ must lie in $(-r < x < r)$. Suppose that α, a zero of $\pi^*_n(z)$, satisfied $|\alpha| \geqq r$. Then, if α were replaced by $\alpha' = \rho\alpha$ where ρ is real and less than but near one, the value of $\mu(n)$ would be

diminished. Also the zeros of $\pi^*_n(z)$ are all simple. To see this suppose that β is a multiple zero of $\pi^*_n(z)$. Let β_1 and β_2 satisfy: 1° $-r < \beta_1 < \beta < \beta_2 < r$; 2° $\max\limits_{\beta_1 < x < \beta_2} (x - \beta_1)/(1 - \beta_1 x) \cdot (\beta_2 - x)/(1 - \beta_2 x) < \mu(n)$; 3° $(\beta - \beta_1)/(1 - \beta\beta_1) = (\beta_2 - \beta)/(1 - \beta\beta_2)$. If the alleged factor, $((z - \beta)/(1 - \beta z))^2$, of $\pi^*_n(z)$ were replaced by $(z - \beta_1)/(1 - \beta_1 z) \cdot (z - \beta_2)/(1 - \beta_2 z)$, the resulting function would be such that its *écart* on the interval $(-r \leqq x \leqq r)$ would be less than $\mu(n)$. This is certainly true for $\beta_1 \leqq x \leqq \beta_2$ by the choice of β_1 and β_2. On the other hand, for those x which satisfy either $-r \leqq x < \beta_1$ or $\beta_2 < x \leqq r$ the readily verified inequality

$$0 < \frac{x - \beta_1}{1 - \beta_1 x} \cdot \frac{x - \beta_2}{1 - \beta_2 x} < \left(\frac{x - \beta}{1 - \beta x}\right)^2 \tag{7.4}$$

prevails. Hence the *écart* on the interval $(-r \leqq x \leqq r)$ of the function replacing $\pi^*_n(z)$ would be less than $\mu(n)$, and this is contrary to the definition of $\mu(n)$.

If $\pi^*_n(z)$ is subjected to the normalization, $\pi^*_n(1) = 1$, then $\pi^*_n(z)$ is real for z real. Since $\pi^*_n(z)$ is of degree n, all the zeros of $d\pi^*_n/dz$ in $|z| < 1$ are accounted for by the points of relative maxima of $|\pi^*_n|$ between successive zeros of π^*_n, there being precisely one zero of $d\pi^*_n/dz$ between successive zeros of π^*_n. The value of $|\pi^*_n(z)|$ at these positions of relative maxima and minima as well as at $z = -r, r$ is $\mu(n)$. Suppose, for example, that $\pi^*_n(r) < \mu(n)$ (it should be observed that $\pi^*_n(r) > 0$ by virtue of the normalization to which $\pi^*_n(z)$ is subjected) and let the zeros of $\pi^*_n(z)$ be denoted by $\alpha_1, \alpha_2, \cdots, \alpha_n$ where

$$-r < \alpha_1 < \alpha_2 < \cdots < \alpha_n < r.$$

If α_n were replaced by α'_n in $\pi^*_n(z)$, where α'_n is real, less than and sufficiently close to α_n, then the resulting function would be such that the maximum of its modulus on the interval $(-r \leqq x \leqq r)$ would be actually less than $\mu(n)$ and this is impossible. A similar argument would apply if $|\pi^*_n(-r)| < \mu(n)$. There remains for consideration the relative maximum of $|\pi^*_n(x)|$ between α_k and α_{k+1} $(k = 1, 2, \cdots, n-1)$. Let γ_k denote the zero of $d\pi^*_n/dz$ lying between α_k and α_{k+1} $(k = 1, 2, \cdots, n-1)$, and suppose that $|\pi^*_n(\gamma_{k_0})| < \mu(n)$ for some index k_0. Replace α_{k_0} and α_{k_0+1} by α'_{k_0} and α''_{k_0+1} respectively in $\pi^*_n(z)$, where α'_{k_0} and α''_{k_0+1} satisfy the following conditions: 1° $\alpha'_{k_0} < \alpha_{k_0} < \alpha_{k_0+1} < \alpha''_{k_0+1}$; 2° α'_{k_0} is so close to α_{k_0} and α''_{k_0+1} is so close to α_{k_0+1} that the maximum of the modulus of the replacing function for $\alpha'_{k_0} < x < \alpha''_{k_0+1}$ is less than $\mu(n)$; 3° the non-Euclidean distance between α'_{k_0} and α_{k_0} is equal to the non-Euclidean distance between α_{k_0+1} and α''_{k_0+1}. The effect of 3° is that

$$(7.5) \qquad 0 < \frac{x - \alpha'_{k_0}}{1 - \alpha'_{k_0}x} \cdot \frac{x - \alpha''_{k_0+1}}{1 - \alpha''_{k_0+1}x} < \frac{x - \alpha_{k_0}}{1 - \alpha_{k_0}x} \cdot \frac{x - \alpha_{k_0+1}}{1 - \alpha_{k_0+1}x}$$

for $-1 < x < \alpha'_{k_0}$ and $\alpha''_{k_0+1} < x < 1$. The function replacing $\pi^*_n(z)$ under this change of zeros would be of modulus less than $\mu(n)$ for $-r \leqq x \leqq r$, which is impossible. To sum up,

THEOREM 7.1. *There exists a* $(1, n)$ *directly conformal transformation of* $|z| < 1$ *onto itself such that the maximum of its modulus on the interval* $(-r \leqq x \leqq r)$ $(0 < r < 1)$ *is equal to* $\mu(n) = \underset{\{\pi_n(z)\}}{\text{g.l.b.}} [\max_{-r \leqq x \leqq r} |\pi_n(x)|]$ *where* $\pi_n(z)$ *denotes a* $(1, n)$ *directly conformal map of* $|z| < 1$ *onto itself. Further, if* $\pi^*_n(z)$ *denotes an extremal function for this problem so normalized that* $\pi^*_n(1) = 1$, *then* $\pi^*_n(z)$ *has the following properties*:

1° $\pi^*_n(z)$ *is real for* z *real*;

2° *the* n *zeros of* $\pi^*_n(z)$ *are all simple and lie in* $(-r < x < r)$;

3° $\pi^*_n(r) = \mu(n)$, $|\pi^*_n(-r)| = \mu(n)$, *the maximum of* $|\pi^*_n(x)|$ *between successive zeros of* $\pi^*_n(z)$ *is equal to* $\mu(n)$, *this value being attained by* $|\pi^*_n(x)|$ *at* r, $-r$, *and the zeros of* $d\pi^*_n/dz$;

4° *the zeros of* $d\pi^*_n/dz$ *are simple, and there is precisely one zero of* $d\pi^*_n/dz$ *between successive zeros of* $\pi^*_n(z)$, *and there are no others.*

It should be remarked that $|\pi^*_n(z)| = \mu(n)$ only at those points enumerated in 3° since $\pi^*_n(z)$ is of degree n. The structure of the Riemannian image of the extended plane with respect to $w = \pi^*_n(z)$ is completely specified by Theorem 7.1. Since the derivative of $\pi^*_n(z)$ at $z = 1$ is positive, it follows from the uniqueness theorem of the theory of the conformal mapping of simply-connected Riemann surfaces that $\pi^*_n(z)$ is uniquely defined by its normalization and its extremal property. Hence

THEOREM 7.2. *The normalized extremal function of Theorem 7.1 is unique and hence is odd if* n *is odd and even if* n *is even. All the extremal functions are given by* $e^{i\theta}\pi^*_n(z)$ $(\theta$ *real*$)$.

There remains to be considered the determination of $\mu(n)$ and $\pi^*_n(z)$. They may be found by the following argument.[7] Consider the function $\Phi(z)$ defined by

$$(7.7) \qquad \Phi(z) \equiv \pi^*_n[B(z; r)].$$

An analysis of $\Phi(z)$ reveals that $\Phi(z) \equiv B[z; \mu(n)]$.

[7] It is readily seen that $\mu(1) = r$ and $\pi^*_1(z) \equiv z$. It will be assumed for the remainder of this section that $n > 1$.

This may be seen by first noting that $B(0;r)=r$ implies $\Phi(0)=\mu(n)$. Second, the structure of $G[\Phi]$ may be readily inferred from the structures of $G[B(z;r)]$ and $G[\pi^*_n]$. From the definition of $B(z;r)$ as $\phi(-z^2;r)$ (5), $G[B(z;r)]$ may be constructed with the aid of a set of simple open disks, S_k, $(k=0,1,2,\cdots;-1,-2,\cdots)$, where each S_k consists of a copy of $|w|<1$ slit along the real axis from -1 to $-r$ and from r to 1. In order to obtain $G[B(z;r)]$ the upper and lower banks of the right slit of S_k are joined to the lower and upper banks respectively of the right slit of S_{k+1}, and the upper and lower banks of the left slit of S_k are joined to the lower and upper banks respectively of the left slit of S_{k-1} for $k=0,\pm 2,\pm 4,\cdots$, the Riemannian domain so obtained being $G[B(z;r)]$. From the descriptive properties of $\pi^*_n(z)$ developed in the present section $G[\pi^*_n]$ may also be constructed. Let σ_λ $(\lambda=1,\cdots,n)$ denote a copy of the open disk, $|w|<1$, slit along the real axis from -1 to $-\mu(n)$ and from $\mu(n)$ to 1. Then $G[\pi^*_n]$ may be formed from the σ_λ as follows. The upper and lower right banks of σ_1 are joined to each other; then the upper and lower left banks of σ_1 are joined to the lower and upper banks respectively of σ_2. If $n=2$, the upper and lower right banks of σ_2 are joined to each other. If $n>2$, then the upper and lower right banks of σ_2 are joined to the lower and upper banks respectively of σ_3, etc. In σ_n one slit remains and the upper and lower banks of this slit are to be joined together. If n is odd, it is the left-hand slit, otherwise the right-hand slit. The validity of this construction is guaranteed by the general properties of $(1,n)$ directly conformal maps of the interior of the unit circle onto itself and the particular ramification properties of $\pi^*_n(z)$ already enumerated.

It follows from this discussion that the Riemannian image of each S_k with respect to $\pi^*_n(z)$ consists of a copy of $G[\pi^*_n]$ with σ_1 slit from $\mu(n)$ to 1 and σ_n slit from $\mu(n)$ to 1 if n is even, or from -1 to $-\mu(n)$ if n is odd. Hence if $\bar{G}^{(k)}[\pi^*_n]$ denotes the Riemannian image of S_k with respect to $\pi^*_n(z)$, $G[\Phi]$ may be constructed in terms of the $\bar{G}^{(k)}[\pi^*_n]$ by connecting the latter along their slits in the following manner. Let it be agreed that the right slit of $\sigma_1^{(0)}$ of $\bar{G}^{(0)}[\pi^*_n]$ ($\sigma_\lambda^{(k)}$ having the connotation for $\bar{G}^{(k)}[\pi^*_n]$ that σ_λ has for $G[\pi^*_n]$) is to be connected to the right slit of $\sigma_1^{(1)}$ and that the as yet unconnected slit of $\sigma_n^{(0)}$ be connected to the corresponding slit of $\sigma_n^{(-1)}$, upper and lower banks being connected with lower and upper banks respectively. The next step is to connect the available slit of $\sigma_n^{(1)}$ to the corresponding slit of $\sigma_n^{(2)}$ and the right slit of $\sigma_1^{(-1)}$ to the right slit of $\sigma_1^{(-2)}$, again with the same type of connection between upper and lower banks. The process is to be continued in this way step-by-step. In this manner it is seen that

$G[\Phi]$ and $G[B(z;\mu(n))]$ are identical. Thus, since $\Phi(0)=B(0;\mu(n))$, $\Phi(z)\equiv B(e^{i\theta}z;\mu(n))$ (θ real). Now $\theta\equiv 0 \pmod \pi$); this may be established by noticing that the zeros of $\Phi(z)$ are to be found from those of $B(z;\mu(n))$ by multiplying those of the latter by $e^{-i\theta}$; all the zeros of $\Phi(z)$ and $B(z;\mu(n))$ being real, the assertion follows. Since $B(z;\mu(n))$ is even, $\Phi(z)\equiv B(z;\mu(n))$.

The following argument determines $\mu(n)$ and the zeros of $\pi^*_n(z)$. Let T_r denote the linear fractional transformation of $|z|<1$ onto itself associated with $B(z;r)$ as T is associated with $B(z;m)$ in **5** and let λ_r denote the associated multiplier (cf. (5.2)); similarly let $T_{\mu(n)}$ and $\lambda_{\mu(n)}$ have the corresponding connotations for $B(z;\mu(n))$. From (7.7) it follows that the cyclic group generated by T_r is a subgroup of the cyclic group generated by $T_{\mu(n)}$. Hence $T_r=T^k{}_{\mu(n)}$ for some positive integer k. Actually $k=n$. This may be seen by noting that $B(z;\mu(n))$ has precisely k simple zeros in the interval $(0<x<T^k{}_{\mu(n)}0)$ or, what is the same, in the interval $(0<x<T_r0)$. Now $B(x;r)$ is monotone decreasing for $(0\leqq x\leqq T_r0)$ and $B(0;r)=r$ and $B(T_r0;r)=-r$. Since the n zeros of $\pi^*_n(z)$ are located in the open interval $(-r,r)$, $\pi^*_n[B(x;r)]\equiv B(x;\mu(n))$ has n simple zeros in $(0<x<T_r0)$. Hence $k=n$. A consequence of this result is that $\lambda_{\mu(n)}=\lambda_r{}^{1/n}$. On reference to (5.6), the value $\mu(n)$ may be obtained at once. It is

$$\mu(n)=\prod_{k=0}^{\infty}\left[\frac{1-\lambda_r{}^{(2k+1)/2n}}{1+\lambda_r{}^{(2k+1)/2n}}\right]^2. \tag{7.8}$$

Also the Blaschke product for $B(z;\mu(n))$ may be written down at once with the aid of (5.4) and (5.5).

The extremal function, $\pi^*_n(z)$, will be "specified" once its zeros are given. These may be found from the identity already employed:

$$B(z;\mu(n))\equiv\pi^*_n[B(z;r)]. \tag{7.9}$$

On the interval $(0<x<T_r0)$, $B(z;\mu(n))$ has n simple zeros. In fact, they are $T_r{}^{(2k-1)/2n}0$ $(k=1,2,\cdots,n)$. Since $B(x;r)$ is $(1,1)$ on the interval $(0\leqq x\leqq T_r0)$, the points $B[T^{(2k-1)/2n}0;r]$ $(k=1,2,\cdots,n)$ are distinct, lie in the interval $(-r<x<r)$ and hence by (7.9) are the zeros of $\pi^*_n(z)$ and there are no others. The argument is complete.

Remark I. This discussion may be paraphrased for the theory of ordinary Tchebycheff polynomials with the starting point of the discussion being the definition of the Tchebycheff polynomial as one which enjoys the extremal property usually associated with it rather than the *ad hoc* definition

$$T_n(x)\equiv\cos(n\arccos x)/2^n.$$

Remark II. It should be noted that the class of competing functions $\{\pi^*_n(z)\}$ is not convex. Hence the useful property of convexity available in the theory of ordinary Tchebycheff polynomials may not be employed here. The full force of the proof is to be found in the representation (7.2) of a competing function, in the associated discussion of the location of the zeros of an alleged extremal function, and in the ramification properties of the mapping it defines.

8. An extremal problem for functions analytic and of modulus less than one in the interior of the unit circle, which vanish at the origin and have an assigned derivative there. The second extremal problem alluded to in **7** is:

PROBLEM. *Let E_α denote the class of functions, $f(z)$, which are analytic and of modulus less than one for $|z| < 1$ and satisfy the conditions: 1° $f(0) = 0$, 2° $f'(0) = \alpha$, where α is an assigned real positive number less than one. Further let $\mu(\alpha) = \underset{f \in E_\alpha}{\text{g.l.b.}} [\underset{-1<x<1}{\text{l.u.b.}} |f(x)|]$. To determine $\mu(\alpha)$ and the corresponding extremal function.*[8]

This problem may be solved in terms of the function $B(z; m)$ by a pr[illegible] choice of m. Since the solution of the present problem calls for many details, the major steps are outlined.

I. *There exists an extremal function for this problem.* This may be inferred from the compactness of the class E_α. Let $\phi(z)$ denote a typical extremal function. Actually it will turn out that there is a unique extremal function.

II. $|\phi(z)|$ *attains* $\mu(\alpha)$ *for some* z *in the open interval of the real axis* $(-1 < x < 1)$.

III. *In fact,* $|\phi(z)|$ *attains* $\mu(\alpha)$ *an infinite number of times on the interval* $(-1 < x < 1)$.

IV. *The extremal function is unique.* From this stage of the solution on, $\phi(z)$ is unambiguous in connotation.

V. $\phi(z)$ *is real for* z *real and is odd.*

VI. *All the zeros of* $\phi(z)$ *are real.*

VII. At this juncture it is convenient to introduce an auxiliary extremal problem. Let $\mu(\alpha, r) = \underset{f \in E_\alpha}{\text{g.l.b.}} [\underset{-r \leqq x \leqq r}{\max} |f(x)|]$, and let it be required to determine properties of the associated extremal function. (The problem of determining $\mu(\alpha, r)$ and the corresponding extremal function explicitly is

[8] Cf. Bernstein [1] where a related problem for entire functions is treated.

difficult but its solution is not essential for the solution of the major problem at hand.) The following properties of the extremal function for this subsidiary problem are listed without proof.[9] (a) *From the compactness of E_a, it follows that there exists an extremal function.* (b) *The extremal function is unique. It defines a $(1, k(r))$ ($k(r)$ a positive integer) directly conformal map of $|z| < 1$ onto itself. Let it be denoted by $\psi(z; r)$.* (c) *$\psi(z; r)$ is real for z real and is odd.* (d) *The zeros of $\psi(z; r)$ are real and simple.* (e) *At most one zero of $\psi(z; r)$ lies to the right of $x = r$ and at most one zero of $\psi(z; r)$ to the left of $x = -r$.* (f) *$\mu(\alpha, r)$ is attained on the real axis by $|\psi(z; r)|$ between successive zeros of $\psi(z; r)$.* (g) *The antecedent of the closed interval $[-\mu(\alpha, r), \mu(\alpha, r)]$ with respect to $w = \psi(z; r)$ is a subset of $(-1 < x < 1)$.* (h) *$\mu(\alpha, r)$ is monotone non-decreasing for $0 < r < 1$ and $\lim_{r \to 1} \mu(\alpha, r) = \mu(\alpha)$. Hence by* IV *$\lim_{r \to 1} \psi(z; r) = \phi(z)$ in the sense of continuous convergence for $|z| < 1$.*

VIII. From VII (g) and (h) it may be proved that *the antecedent of the closed interval $[-\mu(\alpha), \mu(\alpha)]$ with respect to $w = \phi(z)$ is $(-1 < x < 1)$.*

IX. *At each point of $(-1 < x < 1)$ where $|\phi| = \mu(\alpha)$, ϕ' has a zero of order one.*

X. From IX it is concluded that $\phi(z) = B[T^{-\frac{1}{2}}z; \mu(\alpha)]$ *where T has the meaning previously associated with it.*

XI. *$\mu(\alpha)$ is determined in terms of α from the relation $\phi'(0) = \alpha$.*

The proof of I will be omitted inasmuch as the argument is standard. To establish II the auxiliary function, $\eta(z; \beta)$, is introduced, being defined by

$$\eta(z; \beta) \equiv z e^{\beta(z^2+1)/(z^2-1)} \qquad (\beta \text{ real and positive}). \tag{8.1}$$

With this restriction on β, $\eta(z; \beta)$ is analytic and of modulus less than one for $|z| < 1$ and in addition has the property that $\lim_{x \to 1} \eta(x; \beta) = \lim_{x \to -1} \eta(x; \beta) = 0$, where x is assumed real and of modulus less than one. If β is taken equal to $-\log \alpha$, then $\eta(z; \beta) \in E_a$. Let it be assumed for the immediate argument that this is the case. The statement II may be established as follows. Suppose that $\mu(\alpha)$ is not attained by $|\phi(z)|$ on the open interval $(-1 < x < 1)$ and let r be chosen positive and less than, but so close to, one that for x real and satisfying $r \leqq |x| < 1$

$$|\eta(x; \beta)| \leqq \mu(\alpha)/2. \tag{8.2}$$

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[9] The proofs of these properties may be supplied by applying the methods of § 7 of [5]. Since the proofs of the present results are so closely related to those given there, they will be omitted.

Clearly $\max\limits_{-r \leqq x \leqq r} |\phi(x)| < \mu(\alpha)$. Now consider

$$\Phi(z;\lambda) \equiv (1-\lambda)\phi(z) + \lambda\eta(z;\beta) \tag{8.3}$$

for λ real and satisfying $0 \leqq \lambda \leqq 1$. For such λ the function (8.3) belongs to E_α. For λ positive and sufficiently small, $\Phi(z;\lambda)$ satisfies (i) $\max\limits_{-r \leqq x \leqq r} |\Phi(x;\lambda)| < \mu(\alpha)$; (ii) $\operatorname*{l.u.b.}\limits_{r<|x|<1} |\Phi(x;\lambda)| < \mu(\alpha)$ (x real). Hence $\operatorname*{l.u.b.}\limits_{-1<x<1} |\Phi(x;\lambda)| < \mu(\alpha)$ and this contradicts the assumption that $\phi(z)$ is extremal. II follows.

The assertions III and IV may be established by a combined argument. Let $\{x_k\}$ denote the set of points on $(-1 < x < 1)$ for which $|\phi(x_k)| = \mu(\alpha)$ and consider the Pick-Nevanlinna interpolation problem [10] of determining the functions, $g(z)$, which are analytic and of modulus less than one for $|z| < 1$ and satisfy in addition

$$g(x_k) = \phi(x_k), \qquad g(0) = 0 \tag{8.4}$$

for all indices k considered. The following inferences may be drawn from the Pick-Nevanlinna interpolation theory. The case where the set $\{x_k\}$ is finite will be treated first. If the function $g(z)$ is uniquely determined by the interpolation conditions (8.4), then it is rational and defines a $(1, K)$ directly conformal map of $|z| < 1$ onto itself for some positive integer K. This is impossible because then $\phi(z)$ would have the property

$$\lim_{|z|\to 1} |\phi(z)| = 1 \tag{8.5}$$

which is contrary to its definition as an extremal function for the major problem under consideration.

If $g(z)$ is not uniquely determined by (8.4), then the possible values of $g'(0)$ for the class of $g(z)$ considered fill a closed circle containing α. Let ρ (< 1) denote the largest positive number in this circle. Clearly, $\alpha \leqq \rho$. There is a unique member of the class of $g(z)$, say $g_0(z)$, which satisfies the further condition that $g'_0(0) = \rho$; $g_0(z)$ defines a $(1, K)$ directly conformal map of $|z| < 1$ onto itself. The assumption, $\alpha = \rho$, is untenable by the argument used above. If $\alpha < \rho$, take β in $\eta(z;\beta)$ equal to $-\log \alpha/\rho$. Then the function

$$\phi^*(z) \equiv g_0(z)\eta(z;\beta)/z \tag{8.6}$$

belongs to E_α and is such that for the x_k considered

$$|\phi^*(x_k)| < \mu(\alpha). \tag{8.7}$$

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[10] Cf. Walsh [15], R. Nevanlinna [13].

Therefore for λ sufficiently small and positive, $(1-\lambda)\phi(z)+\lambda\phi^*(z)$ belongs to E_a and satisfies

$$(8.8) \qquad \underset{-1<x<1}{\text{l.u.b.}} |(1-\lambda)\phi(x)+\lambda\phi^*(x)| < \mu(\alpha)$$

which contradicts the extremal character of $\phi(z)$. *Therefore an extremal, $\phi(z)$, satisfies* III.

The situation where the set $\{x_k\}$ is infinite remains to be considered. Again, if $g(z)$ is not uniquely determined by (8.4), the set of possible values of $g'(0)$ fills a closed circle, containing α. Let ρ^* (<1) denote the largest positive number in this circle. The possibility that $\alpha<\rho^*$, is excluded by the argument applied to the case where the set $\{x_k\}$ is finite. Hence $\alpha=\rho^*$, and this implies, by the Pick-Nevanlinna theory, that $g(z)$ is uniquely determined by (8.4) and the condition $g'(0)=\alpha$. There remains only the case where $g(z)$ is uniquely determined. At all events the only function analytic and of modulus less than one for $|z|<1$ which satisfies (8.4) and $g'(0)=\alpha$ is $\phi(z)$ itself.

Now suppose that $\phi(z)$ is not unique and let $\phi_1(z)$ and $\phi_2(z)$ denote two distinct extremal functions of class E_a. Clearly, $[\phi_1(z)+\phi_2(z)]/2$ also belongs to E_a, is extremal, and is distinct from ϕ_1 and ϕ_2. Let $\{X_k\}$ denote the infinite set of points on $(-1<x<1)$ for which $|\phi_1+\phi_2|/2=\mu(\alpha)$. At such points $\phi_1=(\phi_1+\phi_2)/2=\phi_2$. Hence by the result of the preceding paragraph

$$(8.9) \qquad \phi_1(z)\equiv\frac{\phi_1(z)+\phi_2(z)}{2}\equiv\phi_2(z)$$

which is impossible; uniqueness is established.

Assertion V is readily proved by noting that along with $\phi(z)$ both $\overline{\phi(\bar{z})}$ and $-\phi(-z)$ are extremal.

Assertion VI may be concluded in the same manner as VII (d). The proof is readily supplied.

Assertion VIII may be established as follows. Suppose that there exists a point z_0 of $|z|<1$, not on the interval $(-1<x<1)$, with $\phi(z_0)$ in the closed interval $[-\mu(\alpha),\mu(\alpha)]$. Then there would exist in $|z-z_0| < \min[|Iz_0|, 1-|z_0|]$ a point ζ such that $\phi(\zeta)$ would lie in the closed interval $[-\mu(\alpha,r),\mu(\alpha,r)]$ for all r less than and sufficiently near one. But then by Hurwitz's theorem, for a given neighborhood of ζ, say $N(\zeta)$, and for r sufficiently near one, $\psi(z;r)$ would attain the value $\phi(\zeta)$ for some z in $N(\zeta)$ and this manifestly contradicts VII (g).

As for IX, the points of $(-1<x<1)$ where $|\phi(x)|=\mu(\alpha)$ are points

where $\phi(x)$ has either a relative maximum or minimum, $\phi(x)$ being considered as a real function of the real variable x. At such points ϕ' has a zero of order one as a consequence of VIII.

Step X is decisive in completing the proof. Let $b(z)$ denote $B(T^{-\frac{1}{2}}z\,;\mu(\alpha))$ and consider the equation

$$b[\psi(z)] = \phi(z), \qquad (\psi(0) = 0). \tag{8.10}$$

Since $b(0) = 0$, it follows that (8.10) defines $\psi(z)$ as a single-valued analytic function of modulus less than one for $|z| < 1$, since $G[\phi]$ is completely ramified with branch points of order one over $\mu(\alpha)$ and $-\mu(\alpha)$. Upon differentiation, (8.10) yields $b'(0)\psi'(0) = \phi'(0)$. By Schwarz's lemma $|\psi'(0)| \leqq 1$. Hence $0 < \alpha = \phi'(0) \leqq b'(0)$. It is impossible that $\alpha < b'(0)$, since then $(\alpha/b'(0))b(z)$ would belong to E_α and yet for real z, $|(\alpha/b'(0))b(z)| \leqq (\alpha/b'(0))\mu(\alpha) < \mu(\alpha)$. This would imply that $\phi(z)$ is not extremal. Therefore since $\alpha = b'(0)$, it follows that $\phi(z) \equiv b(z)$.

There remains to be considered the determination of $\mu(\alpha)$ and the multiplier of T in terms of α (XI). These questions may be treated by much the same methods as those used in **5**. As in that section let T be given by

$$T: \frac{Z-1}{Z+1} = \lambda\,\frac{z-1}{z+1} \qquad (0 < \lambda < 1). \tag{8.11}$$

Then $b(z)$ may be represented by

$$b(z) \equiv z \prod_{k=1}^{\infty} \frac{(T^k 0)^2 - z^2}{1 - (T^k 0)^2 z^2}. \tag{8.12}$$

Hence $\alpha = \prod_{k=1}^{\infty} [T^k 0]^2$. But $(T^k 0)^2 = \left(\dfrac{1-\lambda^k}{1+\lambda^k}\right)^2$. It follows that

$$\alpha = \prod_{k=1}^{\infty} \left(\frac{1-\lambda^k}{1+\lambda^k}\right)^2. \tag{8.13}$$

It is readily seen as in **5** that there exists a unique λ $(0 < \lambda < 1)$ satisfying (8.13). With λ so determined, $\phi(z)$ is given by (8.12). The extremal value, $\mu(\alpha)$, is $b(T^{\frac{1}{2}}0)$. The solution of the problem is complete.

Remark. Related extremal problems may be treated in the same spirit. For example, one may consider the problem of determining

$$\underset{f \in E_\alpha}{\text{g.l.b.}}\ [\underset{0<x<1}{\text{l.u.b.}}\ |f(x)|]$$

and the associated extremal function. The problem treated in this section and the one just alluded to are of interest inasmuch as their solutions yield precise information on the magnitude of a function of class E_α either on a

diameter of the unit circle or a ray issuing from the origin. Thus an easy corollary of the present discussion is

COROLLARY. *If $f(z)$ is analytic and of modulus less than one for $|z| < 1$, and if further $f(0) = 0$, $|f'(0)| = \alpha$ $(0 < \alpha < 1)$, then along each diameter of $|z| < 1$ there exists a point at which $|f| \geqq \mu(\alpha)$. If along any diameter $|f| \leqq \mu(\alpha)$, then $f(z) \equiv e^{i\theta_1}\phi(e^{i\theta_2}z)$, where θ_1 and θ_2 are real constants.*

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THE DENSITIES OF IDEAL CLASSES AND THE EXISTENCE OF UNITIES IN ALGEBRAIC NUMBER FIELDS.*

By Aurel Wintner.

According to a terminology customary in the analytic theory of numbers, the "depth" of an asymptotic law concerning integers or ideals is measured in terms of the (*specific*) *arithmetical* information needed in the proof. Correspondingly, the varying degree of complication in the (*general*) *analytical* theorems of the Tauberian machinery involved is not taken into account by this principle of classification. In this technical sense, the prime number theorem is deeper than Mertens' asymptotic formula, not because a Tauberian theorem relevant for the former requires a proof which happens to be longer than the proof of a Tauberian theorem sufficient for the latter, but merely because the former does, and the latter does not, depend on a certain arithmetical information; namely, on one involving the location of the zeros of $\zeta(s)$.

For the "elementary" laws of algebraic number fields, it is natural to consider a corresponding principle of classification, by isolating the theorems which can be proved without involving the arithmetical fact represented by the existence of a complete system of unities or of an associated lattice. In particular, the Dirichlet-Dedekind theorem, concerning the ergodic distribution of the integral ideals over the various ideal classes of the field, *appears* to be less "deep" (that is to say less arithmetical) than an asymptotic estimate involving such results as Minkowski's theorems on discriminants. However, it is another matter to *prove* that such is the case. In fact, the classical proof of the law of Dirichlet-Dedekind depends on the arithmetical theorem concerning the existence of a lattice.

Thus it might be of interest for algebraists that they are overestimating precisely the "algebraical depth" of the Dirichlet-Dedekind theorem, since *the theorem on the existence of unities* (*lattices*) *is superfluous in the proof for the existence and the equality of the densities of the various ideal classes in an arbitrary* (*finite*) *algebraic number field.*

The resulting proof is not recommended for class-room purposes. It certainly is not simpler than the classical proof. However, it would be about as meaningless to state that the latter is simpler than the former. In fact, one

* Received June 28, 1944.

could say that the classical proof is "simpler" only at the cost of introducing the deep, that is to say arithmetical, circumstance that there exists in the field a complete system of unities, transforming the problem into what Minkowski calls a macroscopic volume-evaluation. But it turns out that, historically, this rôle of the theory of the unities was that of a *deus ex machina* (even though, according to Furtwängler, the theory of algebraic numbers can *to-day* be developed so as to be based on a lattice).

It seems to be of interest that the legitimacy of this methodical point of view is fully supported by the historical development. In fact, when Dirichlet introduced unities into his (quadratic) fields, his sole purpose was the proof of the class-number formulae. But the latter are identical with what results if the Dirichlet-Dedekind relations of an arbitrary field are particularized to the case of a quadratic field. In Kummer's adaptation of Dirichlet's theory to cyclotomic fields, the rôle of the unities appeared in a form less pure, since it was interwoven with the algebraic affairs of Fermat's problem. However, in Dedekind's extension of Dirichlet's work to arbitrary fields, the introduction of the unities served again no useful purpose distinct from the one for which Dirichlet originally devised it in his particular case. For the modern algebraist who cares for generalized theories of unities for the unities' sake, this may not be easy to believe, but he can convince himself by turning over Dedekind's supplements [1] or Minkowski's historical sketch [3].

The ironical twist of historical development, which first makes mathematics devise a theory for a definite purpose and subsequently leads to other concrete ends (this time to Minkowski's geometry of numbers) but ultimately proves that the introduction of the theory was superfluous exactly for the original purpose, is not of course an uncommon occurrence. That it presents itself in case of the Dirichlet-Dedekind theorem, can be proved as follows:

With reference to any ideal class C of a fixed algebraic number field $K = K(\alpha)$, let $f(n; C)$ denote the number of the integral ideals which are contained in C and have a norm not exceeding n, where $n = 1, 2, \cdots$. Then the assertion to be proved is that the ratio $f(n; C)/n$ tends to a finite, non-vanishing limit, say λ; and that, in addition, this limit $\lambda = \lambda_K$ is independent of the choice of C in K.

Let $F(n; C)$ denote the number ($\geqq 0$) of the representations of n as norms of (integral) ideals contained in C. Then it is obvious that

$$f(n; C) = F(1; C) + F(2; C) + \cdots + F(n; C) \tag{1}$$

and that, if $\zeta(s; C)$ denotes the zeta-function of C, the series

$$\zeta(s;C)=\sum_{n=1}^{\infty}F(n;C)/n^s \tag{2}$$

converges in the half-plane $\sigma>1$, where $s=\sigma+it$. It follows that the existence of a positive value $\lambda=\lambda_K$ satisfying

$$(\sigma-1)\zeta(\sigma;C)\to\lambda \quad \text{as} \quad \sigma\to 1+0$$

is a necessary condition for the truth of the assertion. In fact, Dirichlet's Abelian lemma states that, if $a_1, a_2, \cdots$ is any sequence of values possessing a finite mean-value

$$M(a)=\lim_{n\to\infty}(a_1+a_2+\cdots+a_n)/n, \tag{3}$$

then the Dirichlet series

$$\sum_{n=1}^{\infty}a_n/n^s \tag{4}$$

must converge for $\sigma>1$ to a function satisfying

$$(\sigma-1)\sum_{n=1}^{\infty}a_n/n^{\sigma}\to M(a) \quad \text{as} \quad \sigma\to 1+0. \tag{5}$$

On the other hand, the expression on the left of the limit relation (5) can tend to a limit and, what is more, the function represented by (4) in the half-plane $\sigma>1$ can have a simple pole at $s=1$, even though the limit (3) fails to exist. This is shown by a well-known example of Dedekind, rediscovered by Hardy, and subsequently used by Ramanujan for a mistaken proof of the prime number theorem in a manner the illegitimacy of which was verified by Dedekind precisely on this example; cf. [5]. It is true that the coefficients $F(n;C)$ of (2) are non-negative, but this is of no avail, since the restriction $a_n\geqq 0$ is satisfied by Dedekind's counter-example also.

However, as a corollary of Hecke's functional equation, the function $(s-1)\zeta(s;C)$ is regular in the whole plane, and the residue of $\zeta(s;C)$ at the point $s=1$ has a positive value, say $\lambda=\lambda_K$, which is independent of the choice of C in K. This information is available for the purpose at hand. In fact, while Hecke's original proof did make use of the existence of a lattice, he subsequently observed that his functional equation can be proved without any reference to the theory of unities, and such proofs are now available in the literature. For older references cf. [4].

Actually, neither the functional equation nor the regularity of the difference

$$\zeta(s;C)-\lambda/(s-1) \tag{6}$$

in the whole plane will be needed in the sequel. All that will be used is that the function (6), where $\sigma > 1$, goes over into a continuous boundary function on the line $\sigma = 1$.

In fact, this in itself makes applicable Ikehara's theorem, which is a particular case of a purely analytical result concerning Laplace transforms and does not presuppose *any* arithmetical information; cf., e. g., [2]. Ikehara's theorem states that, if $a_1, a_2, \cdots$ is a sequence of real, non-negative values, and if the corresponding Dirichlet series (4) converges in the half-plane $\sigma > 1$ to a function corresponding to which there exists a constant λ such that the difference

$$\sum_{n=1}^{\infty} a_n/n^s - \lambda/(s-1), \tag{6 bis}$$

where $\sigma > 1$, goes over into a continuous boundary function on the line $\sigma = 1$, then

$$a_1 + a_2 + \cdots + a_n \sim \lambda n$$

as $n \to \infty$. Hence, if (4) is identified with (2), it follows from (1) that $f(n; C) \sim \lambda n$.

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SOME RELATIONS BETWEEN THE BEHAVIOR OF A FUNCTION AND THE ABSOLUTE SUMMABILITY OF ITS FOURIER SERIES.*

By Kien-Kwong Chen.

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1. We suppose that $f(t)$ is a periodic function, with period 2π, integrable in the Lebesgue sense. We write

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\},$$

and suppose the Fourier series of $\phi(t)$ to be $\Sigma A_n \cos nt$. Bosanquet[1] has shown that the convergence of $\Sigma \mid A_n \mid$ does not imply that the mean-function

$$[\phi(t)]_1 = (1/t) \int_0^t \phi(u)\,du$$

is of bounded variation in $(0, \pi)$. It has, however,[2] been proved that the later property on $[\phi(t)]_1$ is a consequence of

$$\Sigma \mid A_n \mid \log n < \infty.$$

In the following section we prove that the convergence of $\Sigma \mid a_n \mid$ implies that

$$\int_0^\delta \mid (d/dt)\{(\log 1/t)^{-2}[\phi(t)]_1\}\mid dt < \infty, \quad (0 < \delta < 1).$$

This is the case $\rho(t) = \mid \log t \mid^{-2}$, $A = \delta$ of Theorem 1. The chief object of the present paper is to establish Theorem 2 which includes both the theorems stated above.

2. Theorem 1. *If the Fourier series of $f(t)$ converges absolutely at $t = x$ then the function*

$$\Phi(t) \equiv (\rho(t)/t) \int_0^t \{f(x+t) + f(x-t)\}dt$$

is of bounded variation in $(0, A)$, where $\rho(t)$ denotes any function which is absolutely continuous in $(0, A)$ such that

$$\int_0^A \mid t^{-1}\rho(t) \mid dt < \infty.$$

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* Received May 1, 1944.

[1] L. S. Bosanquet (1).

[2] K. K. Chen (1).

As stated above, this proposition is involved in Theorem 2, but there is a direct proof much simpler than the proof of Theorem 2.

In fact, if $0 < t < A$ then

$$\Phi'(t) \equiv \rho'(t)\Sigma A_n \frac{\sin nt}{nt} + \rho(t)\Sigma A_n \frac{d}{dt}\,\frac{\sin nt}{nt}. \tag{1}$$

Since

$$\int_0^A \left| \rho'(t)\,\frac{\sin nt}{nt} \right| dt \leqq \int_0^A | \rho'(t) |\, dt$$

and

$$\int_0^A \left| \rho(t)\,\frac{d}{dt}\,\frac{\sin nt}{nt} \right| dt \leqq 2 \int_0^A | \rho(t) t^{-1} |\, dt,$$

it follows from equation (1) that

$$\int_0^A | \Phi'(t) |\, dt \leqq \left(\int_0^A | \rho'(t) |\, dt + 2 \int_0^A | \rho(t) t^{-1} |\, dt \right) \Sigma\, | A_n |.$$

This proves the theorem.

3. Write now $\sigma_n{}^\alpha$ for the n-th Cesàro mean of order α of the series ΣA_n. The series ΣA_n is said to be super-absolutely summable (C, α) to the sum S with respect to the sequence $l = \{l_n\}$ $(0 < l_n < l_{n+1})$ or summable $l \mid C, \alpha \mid$ to S, if $\sigma_n{}^\alpha \to S$ and the series

$$\Sigma l_n \mid \sigma_n{}^\alpha - \sigma_{n-1}{}^\alpha \mid$$

is convergent. We denote this mode of summability by the equation

$$\Sigma A_n = S(l \mid C, \alpha \mid). \tag{2}$$

We also write, for an integrable function $\chi(u)$,

$$[\chi(t)]_\alpha = \alpha t^{-\alpha} \int_0^t (t - u)^{\alpha-1} \chi(u)\, du \qquad (\alpha > 0)$$

as the mean function of order α of $\chi(u)$ on $(0, t)$.

The theorem on the summability $\{\log n\} \mid C, 0 \mid$ stated in **1** has been extended as follows:[3] if (2) holds true for a sequence $l = \{(\log n)^p\}$ $(p \geqq 1)$ and an order $\alpha > -1$, then the function $| \log t |^{p-1} \cdot t^\lambda [t^{-\lambda} \phi(t)]_{1+\alpha}$ is of bounded variation in $(0, \pi)$, provided that $\lambda < 1$, $\lambda \leqq 1 - (\alpha)$, where (α) denotes $\alpha - [\alpha]$, the fractional part of α. The proof depends upon a number of lemmas, some of which are recalled in the next section as they are relevant to the investigations of the present paper.

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[3] K. K. Chen (1).

4. LEMMA 1. *Let* $b \geqq \alpha > -1$, $\lambda < 1$, $\lambda \leqq 1 - (\alpha)$, $0 < t \leqq \pi$, *and write*

$$(\beta)_n = \frac{(\beta+1)(\beta+2)\cdots(\beta+n)}{n!}, \qquad C(x) = \int_0^1 u^{-\lambda}(1-u)^b \cos(xu)\,du;$$

then the series

$$\sum_{\mu=\nu}^{\infty} (\alpha)_\mu \sum_{n=\mu}^{\infty} (-2-\alpha)_{n-\mu} C(nt)$$

converges to a function $F_\nu(t)$ *such that*

$$F_\nu(t) = O(\nu t)^{-p} \ (p\text{: a positive constant}), \qquad tF'_\nu(t) = O(1).$$

Moreover, if $\nu t \leqq 1$, *then*

$$F_\nu(t) = B(1-\lambda, 1+\alpha) + O(\nu t)^{\alpha+1}, \qquad F'_\nu(t) = O(\nu).$$

LEMMA 2. *Under the conditions of Lemma* 1, *we have*

$$\int_0^1 u^{-\lambda}(1-u)^b \phi(ut)\,du = \sum_{\nu=0}^{\infty} (\sigma_\nu{}^\alpha - \sigma_{\nu-1}{}^\alpha) F_\nu(t) \qquad (\sigma_{-1}{}^\alpha = 0),$$

provided that $\Sigma \mid \sigma_n{}^\alpha - \sigma_{n-1}{}^\alpha \mid$ *converges.*

5. The method we have chosen for the purpose of establishing the theorem stated in **3** may be generalized. Indeed, we can prove the following theorem.

THEOREM 2. *Let* $\rho(t)$ *be a monotone function of* t *in* $(0, \pi)$, *satisfying the conditions*

$$(3) \qquad 0 < \rho(t) = O(|\log t|^A), \qquad \rho(t) = o(L(t)),$$

as $t \to 0$, *where* A *is a constant, and* $L(t) = \int_t^\pi \rho(u) u^{-1} du$. *If the Fourier series of* $f(t)$, *at the point* $t = x$, *is summable* $l \mid C, \alpha \mid$ *for an order* $\alpha > -1$ *to the sum* 0, *where* $l_n = L(1/n)$, $(n = 1, 2, 3, \cdots)$, *then the function*

$$\Psi(t) = \rho(t) t^\lambda [t^{-\lambda}\phi(t)]_{1+\alpha} \qquad (\lambda < 1, \ \lambda \leqq 1 - (\alpha))$$

is of bounded variation in $(0, \pi)$.

Before proving this theorem it is convenient to give first the proof of the following lemma.

6. LEMMA 3. *If the derivative* $\rho'(t)$ *of* $\rho(t)$ *exists and is continuous for* $0 < t \leqq \pi$, *then the conditions* (3) *involve*

$$(4) \qquad (1/t)\int_0^t \rho(u)du = o(L(t)), \qquad (t \to 0).$$

Firstly, we prove that the relation

$$(5) \qquad \int_0^t \rho'(u)u^p du = o(t^p L(t))$$

holds true for every positive p, as $t \to 0$.

In fact, in virtue of $t^p\rho(t) = o(1)$, integration by parts gives

$$(6) \qquad \int_0^t \rho'(u)u^p du = \rho(t)t^p - \int_0^t \rho(u)\cdot pu^{p-1}du.$$

Using the relation $\rho(t) = o(L(t))$, we have

$$(7) \quad \int_0^t \rho(u)\cdot pu^{p-1}du = o(\int_0^t L(u)\cdot pu^{p-1}du) = o(L(t)t^p + \int_0^t u^{p-1}\rho(u)du),$$

in view of the fact that $L(t)t^p = t^p \int_t^\pi o(u^{-1-p/2})du = o(1)$.

Since $\rho(u) > 0$, it follows from (7) that

$$\int_0^t \rho(u)\cdot pu^{p-1}du = o(L(t)t^p).$$

On combining this relation with (6), we obtain (5), by using $\rho(t) = o(L(t))$ again.

On integration by parts, we obtain

$$(1/t)\int_0^t \rho(u)du = \rho(t) - (1/t)\int_0^t u\rho'(u)du = o(L(t)),$$

by appealing to (5) and (3). This establishes the lemma.

7. Lemma 4. *If the Fourier series of $f(t)$, at $t = x$, is summable $l\,|\,C, \alpha\,|$ $(\alpha > -1)$ to the sum 0, then there is a sequence $\{t_n\}$ such that*

$$0 < t_n \to 0, \qquad \lim \Psi(t_n) = 0,$$

where $l = \{l_n\}$, $l_n = L(1/n)$, $L(t) = \int_t^\pi \rho(u)u^{-1}du$, and the functions $\rho(t)$ and $\Psi(t)$ satisfy the conditions of Theorem 2.

In fact, by hypothesis, the positive function

$$\epsilon(t) = \rho(t)/L(t)$$

is $o(1)$, as $t \to 0$. We shall use the function

(8) $$\rho = \rho(t) = \max\left(\tfrac{3}{4}, 1 + 1/2\epsilon(t) \log t\right), \qquad (0 < t < 1).$$

In virtue of the monotony of $\rho(t)$, the derivative $\rho'(t)$ has a definite sign for small t. If $\rho'(t) \leqq 0$, we have

$$L(t^p)/\rho(t) = 1/\epsilon(t) - (1/\rho(t)) \int_t^{t^p} (\rho(u)/u)\,du$$
$$> 1/\epsilon(t) - 1/\rho(t) \cdot \rho(t) \log t^p/t \geqq 1/2\epsilon(t),$$

by (8). Or, if t is small enough,

$$\rho(t)/L(t^p) < 2\epsilon(t).$$

This result is also true, when $\rho'(t) > 0$. Indeed, if $L(0)$ is a convergent integral, then it follows from

$$\lim_{t\to 0} L(t)/L(t^p) = 1$$

that

(9) $$\rho(t)/L(t^p) = \epsilon(t)L(t)/L(t^p) < 2\epsilon(t),$$

for small t. On the contrary, if $L(0) = \infty$, we have

$$1 \leqq \limsup_{t\to 0} L(t)/L(t^p) = \limsup_{t\to 0} [dL(t)/dt]/[dL(t^p)/dt]$$
$$= \limsup_{t\to 0} \rho(t)/p\rho(t^p) \leqq 1/p \leqq 4/3.$$

From this (9) follows, if t is sufficiently small.

Since the summability $l \mid C, \alpha \mid$ of ΣA_n implies the convergence of $\Sigma \mid \sigma_n{}^{\alpha} - \sigma_{n-1}{}^{\alpha} \mid$, we have by Lemma 2,

$$\Phi(t) \equiv \Psi(t)/\rho(t) = t^{\lambda}[t^{-\lambda}\phi(t)]_{1+\alpha}$$
$$= \Big(\sum_{n \leqq t^{-1}} + \sum_{n > t^{-1}}\Big)(\alpha + 1)(\sigma_n{}^{\alpha} - \sigma_{n-1}{}^{\alpha})F_n(t) = \Phi_1(t) + \Phi_2(t).$$

By Lemma 1, we have $F_n(t) = O(1)$. Hence

$$\Phi_2(t) = O\Big(\sum_{n > t^{-1}} \mid \sigma_n{}^{\alpha} - \sigma_{n-1}{}^{\alpha} \mid\Big) = O\Big([1/L(t)] \sum_{n > t^{-1}}^{\infty} l_n \mid \sigma_n{}^{\alpha} - \sigma_{n-1}{}^{\alpha} \mid\Big).$$

It follows that

(10) $$\rho(t)\Phi_2(t) = o(1),$$

as $t \to 0$, since $\rho(t) = o(L(t))$ and ΣA_n is summable $l \mid C, \alpha \mid$.

Writing $[t^{-1}] = T$, we have to consider $\Phi_1(t)$ wherein

$$F_n(t) = B(1-\lambda, 1+\alpha) + O(\nu t)^{\alpha+1},$$

by Lemma 1. Consequently, we may write

$$(11) \quad \Phi_1(t) = (\alpha+1)B(1-\lambda, 1+\alpha)\sigma_T{}^{\alpha} + \sum_{n=0}^{T} (\sigma_n{}^{\alpha} - \sigma_{n-1}{}^{\alpha}) O(nt)^{\alpha+1}.$$

Remembering that $\sigma_n{}^{\alpha} = o(1)$, we have

$$\sigma_T{}^{\alpha} = \sum_{n=T+1}^{\infty} (\sigma_{n-1}{}^{\alpha} - \sigma_n{}^{\alpha}).$$

Accordingly, $| \rho(t)\sigma_T{}^{\alpha} | \leqq \epsilon(t) \sum_{n>T} l_n | \sigma_n{}^{\alpha} - \sigma_{n-1}{}^{\alpha} |$. Hence we obtain

$$(12) \qquad \rho(t)\sigma_T{}^{\alpha} = o(1).$$

In virtue of the relations $\alpha + 1 > 0$, $T\nu \leqq 1$, we have, on writing $M = M(t)[t^{-p}]$,

$$(13) \quad \sum_{n=0}^{T} | \sigma_n{}^{\alpha} - \sigma_{n-1}{}^{\alpha} | (nt)^{\alpha+1} \leqq (Mt)^{\alpha+1} \sum_{n=0}^{M} | \sigma_n{}^{\alpha} - \sigma_{n-1}{}^{\alpha} | + \sum_{n=M+1}^{T} | \sigma_n{}^{\alpha} - \sigma_{n-1}{}^{\alpha}$$

It follows from (8) that

$$t^{(1-p)(1+\alpha)} = \max\{t^{(1+\alpha)/4}, \exp(-(1+\alpha)/2\epsilon(t))\},$$

so that

$$(14) \quad T_1(t) \leqq t^{(1-p)(1+\alpha)} \sum | \sigma_n{}^{\alpha} - \sigma_{n-1}{}^{\alpha} | = O(t^{(1+\alpha)/4}) + O(\exp(-(1+\alpha)/2\epsilon(t))).$$

On differentiating the equation $\rho(t)/\epsilon(t) = L(t)$ with respect to t, and multiplying by $\epsilon(t)/\rho(t)$, and then integrating the equality obtained from t to 1, we get

$$\rho(t)/\epsilon(t) = C \exp\left(\int_t^1 (\epsilon(u)/u)\,du\right.$$

with $C = L(1)$. Take a positive number K small enough such that

$$(15) \qquad K\rho(t) \leqq | \log t |^{A},$$

and set $G = A + 1$. If there exists a positive number t_0 such that

$$\epsilon(t) \log 1/t \geqq G$$

for $0 < t \leqq t_0$, then we have

$$\rho(t)/\epsilon(t) \geqq C \exp(\int_t^{t_0} (\epsilon(u)/u)\,du$$
$$\geqq C \exp(G \log\,(\log t/\log t_0) = C(\log t/\log t_0)^G.$$

On combining this result with (15), we obtain

$$(16) \qquad 1/\epsilon(t) \geqq (KC/|\log t_0|^G)\,|\log t| \qquad (0 < t \leqq t_0).$$

Writing

$$\gamma = \min((1+\alpha)/4,\ (1+\alpha)KC/2\,|\log t_0|^G),$$

it follows from (14) and (16) that

$$(17) \qquad \rho(t)T_1(t) = O(\rho(t)t^\gamma) + O(\exp(-\gamma\,|\log t|))$$
$$= O(t^\gamma\,|\log t|^A) = o(1).$$

On the contrary, if there exists no t_0 such as assumed above, then there is a sequence of positive numbers t_ν, $t_\nu \to 0$, satisfying

$$\epsilon(t_\nu) \log 1/t_\nu < G;$$

and (14) gives

$$(18) \quad \rho(t_\nu)T_1(t_\nu) = O(\rho(t_\nu)t^{(\alpha+1)/2}) + O(\rho(t_\nu))\exp(-(1+\alpha)/2G \log 1/t_\nu)$$
$$= O(t_\nu{}^{(\alpha+1)/2G}\,|\log t_\nu|^A) = o(1).$$

By (17), the relation (18) holds good for any case.

Finally, it follows from

$$T_2(t) = \sum_{M+1}^{N} |\sigma_n{}^\alpha - \sigma_{n-1}{}^\alpha| \leqq (1/L(t^p)) \sum_{M+1}^{N} l_n\,|\sigma_n{}^\alpha - \sigma_{n-1}{}^\alpha|$$

and $\rho(t) < 2L(t^p)\epsilon(t)$ that

$$(19) \qquad \rho(t)T_2(t) < 2\epsilon(t) \sum l_n\,|\sigma_n{}^\alpha - \sigma_{n-1}{}^\alpha| = o(1).$$

Combining the results (10), (11), (12), (13), (18) and (19), we arrive at the relation

$$\Psi(t_\nu) = \rho(t_\nu)\Phi(t_\nu) = \rho(t_\nu)\Phi_1(t_\nu) + \rho(t_\nu)\Phi_2(t_\nu) = o(1).$$

This proves the lemma.

8. We can now prove Theorem 2. It follows from Lemmas 1 and 2 that the relation

$$\Phi'(t) = (1+\alpha) \sum_{n=0}^{\infty} (\sigma_n{}^\alpha - \sigma_{n-1}{}^\alpha) F'_n(t) \tag{20}$$

holds true for $0 < t \leqq \pi$, since $F'_n(t) = O(t^{-1})$ and the series

$$\sum | \sigma_n{}^\alpha - \sigma_{n-1}{}^\alpha |$$

is convergent.

Using Lemma 1, we have

$$\begin{aligned}\int_0^\pi \rho(t) \mid F'_n(t) | \, dt &= \int_0^{1/n} \rho(t) \mid F'_n(t) | \, dt + \int_{1/n}^\pi \rho(t) \mid F'_n(t) | \, dt \\ &= \int_0^{1/n} \rho(t) O(n) dt + \int_{1/n}^\pi \rho(t) t^{-1} dt.\end{aligned}$$

The second term is l_n. And the first term is by Lemma 3, $o(l_n)$ as $n \to \infty$. Hence

$$\int_0^\pi \rho(t) \mid F'_n(t) | \, dt \sim l_n. \tag{21}$$

By (20) and (21), there is a constant K satisfying

$$\int_0^\pi | \rho(t) \Phi'(t) | \, dt \leqq K \sum_{n=0}^{\infty} l_n \mid \sigma_n{}^\alpha - \sigma_{n-1}{}^\alpha |. \tag{22}$$

Now, let t_n be the numbers given by Lemma 4; then $\Psi(t_n) = o(1)$. Integration by parts gives

$$\begin{aligned}\int_0^\pi \rho'(t) \mid \Phi(t) | \, dt &= \lim_{n \to \infty} \int_{t_n}^\pi \rho'(t) \mid \Phi(t) | \, dt \\ &= \rho(\pi) \mid \Phi(\pi) | - \int_0^\pi \rho(t) (d/dt) \mid \Phi(t) | \, dt.\end{aligned}$$

In virtue of the monotony of $\rho(t)$, the above relation implies that

$$\int_0^\pi | \rho'(t) \Phi(t) | \, dt \leqq \rho(\pi) \mid \Phi(\pi) | + \int_0^\pi \rho(t) \mid \Phi'(t) | \, dt.$$

Hence, by (22),

$$\int_0^\pi | \rho'(t) \Phi(t) | \, dt \leqq \rho(\pi) \mid \Phi(\pi) | + K \sum l_n \mid \sigma_n{}^\alpha - \sigma_{n-1}{}^\alpha |. \tag{23}$$

We have $\Psi'(t) = \rho(t)\Phi'(t) + \rho'(t)\Phi(t)$. Hence, by (22) and (23), we obtain

$$\int_0^\pi | \Psi'(t) | \, dt \leqq \rho(\pi) \mid \Phi(\pi) | + 2K \sum l_n \mid \sigma_n{}^\alpha - \sigma_{n-1}{}^\alpha |.$$

This establishes Theorem 2.

9. In his paper "The absolute Cesàro summability of a Fourier series,"[4] Bosanquet asserts nothing concerning the behavior of the function

$$[\phi(t)]_{1+\alpha} = (1+\alpha)t^{-1-\alpha}\int_0^t (t-u)^\alpha \phi(u)\,du,$$

when the Fourier series ΣA_n is summable $|C,\alpha|$ $(\alpha \geqq 0)$. It should be observed that the summability $|C,\alpha|$, is equivalent to the summability

$$\{L(1/n)\}\,|C,\alpha| \qquad (\alpha > -1)$$

in the case where $L(t)$ converges to a finite number, as $t \to 0$. Accordingly we can infer from Theorem 2 the following proposition.

THEOREM 3. *If the Fourier series of $f(t)$, at $t = x$, is summable $|C,\alpha|$ $(\alpha > -1)$, then the function*

$$\rho(t)t^\lambda[t^{-\lambda}\phi(t)]_{1+\alpha} \qquad (\lambda < 1,\ \lambda \leqq 1-(\alpha))$$

is of bounded variation in $(0,\pi)$, where $\rho(t)$ is any function which is absolutely continuous in $(0,\pi)$ such that

$$\int_0^\pi |t^{-1}\rho(t)|\,dt < \infty,$$

and where $2\phi(t)$ denotes $f(x+t)+f(x-t)$.

In fact, the relation (21) is now reduced to

$$\int_0^\pi |\rho(t)F'_n(t)|\,dt = O(1).$$

Consequently, (22) becomes

$$\int_0^\pi |\rho(t)\Phi'(t)|\,dt \leqq K\,|\sigma_n{}^a - \sigma_{n-1}{}^a|. \tag{24}$$

We next have

$$\int_0^\pi |\rho'(t)F_n(t)|\,dt = O(1),$$

by the relations $F_n(t) = O(1)$ and $\int_0^\pi |\rho'(t)|\,dt < \infty$. It follows that

$$\int_0^\pi |\rho'(t)\Phi(t)|\,dt \leqq K'\sum |\sigma_n{}^a - \sigma_{n-1}{}^a|, \tag{25}$$

K' being a constant.

[4] L. S. Bosanquet (2).

On combining the results (24) and (25), we obtain

$$\int_0^{\pi} |(d/dt)\{\rho(t)\Phi(t)\}|\, dt \leqq (K + K') \sum | \sigma_n{}^{a} - \sigma_{n-1}{}^{a} |.$$

The theorem is thus proved.

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A GENERALIZATION OF HARDY'S THEOREM WITH AN APPLICATION TO THE ABSOLUTE SUMMABILITY OF FOURIER SERIES.*

By Kien-Kwong Chen.

1. Suppose that $f(x)$ is integrable in the Lebesgue sense on any finite interval, and that $f(x+2\pi)=f(x)$. Then $[\phi(t)]_\alpha$, the mean function of f, at the point x, of order α, may be defined by the equation

$$[\phi(t)]_\alpha=\alpha t^{-\alpha}\int_0^t (t-u)^{\alpha-1}\phi(u)\,du$$

where $\alpha>0$, $2\phi(t)=f(x+t)+f(x-t)$. If $[\phi(t)]_1$ is of bounded variation in $(0,\pi)$, then x is said to be a point of de la Vallée-Poussin for f. At such a point, the Fourier series of f is summable $|C,\alpha|$ for every $\alpha>1$.[1] The point x is said to be a Dini point of f, if $t^{-1}\phi(t)$ is integrable in the Lebesgue sense on $(0,\pi)$.

Hardy[2] has proved that a Dini point is a point of de la Vallée-Poussin. Evidently, the integrability of $t^{-1}\phi(t)$ depends only on the behavior of the function f in the neighborhood of the point x. Bosanquet and Kestelman[3] have shown that the summability $|C,1|$ for a Fourier series at a given point is not a local property of the function under consideration. Therefore, at a Dini point of f, the Fourier series is not necessarily summable $|C,1|$.

Theorem 1 and Theorem 4 of this paper are generalizations of Hardy's theorem. From these propositions, we derive Theorem 2 and Theorem 3. Direct proofs of the latter theorems are also given.

2. We begin with the following

Lemma 1. *If $l(t)\in L(0,\pi)$, then*

$$h(t)=t^{-3}\int_0^t\int_0^u vl(v)\,dv\,du\in L(0,\pi).$$

To prove this proposition, we may assume that $l(t)$ is non-negative. Then we have

* Received August 17, 1943.

[1] L. S. Bosanquet 1.

[2] G. H. Hardy 1.

[3] L. S. Bosanquet and H. Kestelman 1.

$$h(t) = (1/t^2) \int_0^t wl(w)\,dw - \int_0^t w^2 l(w)\,dw \leqq (1/t^2) \int_0^t wl(w)\,dw$$

and the result is a consequence of the following lemma:

LEMMA 2. *If $l(t) \in L(0,\pi)$, then*

$$t^{-2} \int_0^t ul(u)\,du \in L(0,\pi).$$

For the proof, we can assume that $l(t) \geqq 0$. It follows from Hardy's theorem that the function

$$L(t) = (1/t) \int_0^t ul(u)\,du$$

is of bounded variation in $(0,\pi)$. This involves the existence of the limit $L(+0)$. Integrating by parts,

$$\int_0^\pi l(t)\,dt = L(\pi) - L(0) + \int_0^\pi t^{-2} \int_0^t ul(u)\,du\,dt.$$

This establishes Lemma 2.

3. *If the Fourier series of f is absolutely summable C at the point x, then $t^{-\lambda}\phi(t) \in L(0,\pi)$ for every $\lambda < 1$, but not for $\lambda = 1$.* The proof of this theorem will be given in another paper.

Let us now assume that the integral

$$\chi(t) = \int_{+0}^\pi \phi(u) u^{-1}\,du$$

exists in the Cauchy sense, i. e. the limit $\lim_{\epsilon \to +0} \int_\epsilon^\pi$ exists as a finite number. Assume further that the function $l(t) = t^{-1}\chi(t)$ is integrable in the Lebesgue sense on $(0,\pi)$. Then, the mean function $[\phi(t)]_2$ of the second order is of bounded variation in $(0,\pi)$.

In fact we have

$$[\phi(t)]_2 = (2/t^2) \int_0^t \int_0^u v \cdot \phi(v) v^{-1}\,dv\,du$$

$$= (2/t^2) \int_0^t u\chi(u)\,du - (2/t^2) \int_0^t \int_0^u \chi(v)\,dv\,du.$$

Integration by parts yields

$$\int_0^t u\chi(u)\,du = t \int_0^t \chi(u)\,du - \int_0^t \int_0^u \chi(v)\,dv\,du.$$

On differentiation, we obtain

$$(d/dt)[\phi(t)]_2 = (2/t)\chi(t) - (6/t^2)\int_0^t \chi(u)\,du + (8/t^3)\int_0^t \int_0^u \chi(v)\,dv\,du.$$

Hence, by Lemma 1 and Lemma 2, we have

$$\int_0^\pi |(d/dt)[\phi(t)]_2|\,dt < \infty.$$

The following theorem has thus been established:

THEOREM 1. *If $t^{-1}\chi(t) \in L(0,\pi)$, then $[\phi(t)]_2$ is of bounded variation in $(0,\pi)$.*

A combination of Theorem 1 and a Theorem of Bosanquet[4] gives

THEOREM 2. *If $t^{-1}\chi(t) \in L(0,\pi)$, then the Fourier series of f at the point x is summable $|C,\alpha|$ for every $\alpha > 2$.*

4. Next, we shall give a direct proof of Theorem 2. This is based on the following two lemmas.

LEMMA 3. *Suppose that $a < 1$, $b > 1$ and $h(t) \geqq 0$. Then the existence of the integral $\int_0^t h(t)t^{-1}dt$ implies the convergence of the following two series*

$$\sum_1^\infty n^{-a}\int_0^{1/n} t^{-a}h(t)\,dt \quad \text{and} \quad \sum_1^\infty n^{-b}\int_{1/n}^1 t^{-b}h(t)\,dt.$$

In fact, we have

$$u_n = n^{-a}\int_0^{1/n} t^{-a}h(t)\,dt = \int_0^{1/n} \min\{(nt)^{-a}, (nt)^{-b}\}h(t)\,dt$$

$$v_n = n^{-b}\int_{1/n}^1 t^{-b}h(t)\,dt = \int_{1/n}^1 \min\{(nt)^{-a}, (nt)^{-b}\}h(t)\,dt.$$

Writing

$$H(t) = \sum_{n=1}^\infty t\min\{(nt)^{-a}, (nt)^{-b}\}$$

we have

$$\Sigma u_n \leqq \int_0^1 H(t)t^{-1}h(t)\,dt, \qquad \Sigma v_n \leqq \int_0^1 H(t)t^{-1}h(t)\,dt.$$

On account of the hypotheses $h(t) \geqq 0$ and $t^{-1}h(t) \in L(0,1)$, the convergence of the two series Σu_n and Σv_n follows from the boundedness of $H(t)$. We have

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[4] L. S. Bosanquet 1.

$$H(t) = t^{1-a} \sum_{nt\leqq 1} n^{-a} + t^{1-b} \sum_{nt>1} n^{-b}$$
$$< \frac{t^{1-a}}{1-a}(1/t)^{1-a} + \frac{t^{1-b}}{b-1}(1/t)^{1-b} + 2 = \frac{2-a}{1-a} + \frac{b}{b-1}\,.$$

This completes the proof of the lemma.

LEMMA 4. *Write $g^{\alpha}(n,t)$ for the n-th Cesàro mean of order α of $2\pi^{-1}\sin nt$ $(n = 0, 1, 2, \cdots)$; then for $\alpha > 0$, $0 < t < \pi$, $n > 0$, $\lambda \geqq 0$*

$$|(d/dt)^{\lambda} g^{\alpha}(n,t)| \leqq Kn^{\lambda}(1+nt)^{-\mu}$$

where $\mu = \min(\alpha, 1+\lambda)$, K being a constant.

This is a known theorem.[5]

Let $2 < \beta < 3$, and write $r_n{}^{\beta}$ and $\sigma_n{}^{\beta}$ for the n-th Cesàro mean of order β of the sequences

$$\{\tfrac{1}{2}a_0 + a_1 + \cdots + a_n\} \quad \text{and} \quad \{na_n\}$$

respectively, where a_n $(n = 0, 1, \cdots)$ denote the Fourier coefficients of the function $\phi(t)$. If the series

$$\sum_{n=0}^{\infty} (\sigma_n{}^{\beta} - \sigma_{n-1}{}^{\beta})$$

with $\sigma_{-1}{}^{\beta} = 0$, converges absolutely to the sum S, then we write

$$\tfrac{1}{2}a_0 + \sum_{1}^{\infty} a_n = s \mid C, \beta \mid.$$

An elementary calculation gives

$$\sigma_n{}^{\beta} - \sigma_{n-1}{}^{\beta} = n^{-1} r_n{}^{\beta}.$$

We have to show that the convergence of $\Sigma n^{-1} \mid r_n{}^{\beta} \mid$ follows from

$$t^{-1}\chi(t) \in L(0,\pi).$$

Since

$$na_n = (2/\pi) \int_0^{\pi} \phi(t)\,(d/dt) \sin nt dt,$$

we have

$$r_n{}^{\beta} = (2/\pi) \int_0^{\pi} \phi(t)\,(d/dt) g^{\beta}(n,t)\,dt.$$

Now, on integration by parts,

[5] N. Obrechkoff 1. Cf. also L. S. Bosanquet 1, p. 519.

$$\int_0^\pi \phi(t)(d/dt)g^\beta(n,t)\,dt$$
$$= [t\chi(t)(d/dt)g^\beta(n,t)]_0^\pi - \int_0^\pi \chi(t)(d/dt)\{t(d/dt)g^\beta(n,t)\}dt.$$

This is equal to

$$O(n^{-1}) + \int_0^\pi \chi(t)O(1+nt)^{-2}dt + n^2\int_0^\pi t\chi(t)O(1+nt)^{-\beta}dt,$$

in view of Lemma 4. Hence there is a constant K such that

$$\Sigma\,|\,r_n{}^\beta/n\,| \leqq K\Sigma(1/n^2) + K\Sigma\int_0^{1/n}|\,\chi(t)\,|\,dt + K\Sigma n^{-2}\int_{1/n}^\pi t^{-2}\,|\,\chi(t)\,|\,dt$$
$$+ K\Sigma n\int_0^{1/n} t\,|\,\chi(t)\,|\,dt + K\Sigma n^{1-\beta}\int_{1/n}^\pi t^{1-\beta}\,|\,\chi(t)\,|\,dt.$$

The convergence of the last four series is, by Lemma 3, a consequence of the integrability of $t^{-1}\,|\,\chi(t)\,|$. This establishes the relation

$$\tfrac{1}{2}a_0 + \sum_1^\infty a_n = S\,|\,C,\beta\,|,$$

for $2 < \beta < 3$. The proof of Theorem 2 may be completed by the following theorem of Kogbetliantz.[6]

LEMMA 5. *If $\alpha > 0$, then $\Sigma a_n = S\,|\,C,\alpha\,|$ implies $\Sigma a_n = S\,|\,C,\beta\,|$ for $\beta > \alpha$.*

5. Suppose that $l(t) \in L(o,\pi)$. Write

$$h_n(t) = t^{-1-n}\int_0^t\int_0^{t_{n-1}}\int_0^{t_{n-2}}\cdots\int_0^{t_1} t_0 l(t_0)\,dt_0dt_1\cdots dt_{n-1}.$$

Lemma 1 can be generalized as follows:

LEMMA 6. *The relation $l(t) \in L(0,\pi)$ implies $h_n(t) \in L(0,\pi)$, for every positive integer.*

In fact, integration by parts gives

$$h_n(t) = h_{n-1}(t) - t^{-1-n}\int_0^t u^n h_{n-2}(u)\,du.$$

To obtain the relations $h_n(t) \in L(0,\pi)$ $(n = 1, 2, \cdots)$, we can assume that

[6] E. Kogbetliantz 1. See also E. Kogbetliantz 2.

$\bar{l}(t) \geqq 0$ so that $h_n(t) \leqq h_{n-1}(t)$, for $n = 2, 3, \cdots$. Then, Lemma 2 completes the proof. The arguments developed in the preceding article can be used to extend Theorem 2.

THEOREM 3. *If the repeated Cauchy integral*

$$l(t) \equiv (1/t) \int_{+0}^{t} (1/t_{m-1}) \int_{+0}^{t_{m-1}} (1/t_{m-2}) \int_{+0}^{t_{m-2}} \cdots (1/t_1) \int_{+0}^{t_1} (\phi(t_0)/t_0)\, dt_0 dt_1 \cdots$$

exists as a function integrable in the Lebesgue sense on the interval $(0, \pi)$, *then the Fourier series of* f, *at the point* x, *is absolutely summable* $(C, m + 1 + \epsilon)$ *for every positive* ϵ.

In fact, let $\beta > m + 1$; using Lemma 4, integration by parts gives

$$r_n{}^\beta = (-1)^m \int_0^\pi l(t)\,(t(d/dt))^{m+1} g^\beta(n, t)\, dt + O(1/n)$$

$$= \int_0^\pi l(t) \sum_{k=0}^{m} (1 + nt)^{-1-k} O(nt)^k dt$$

$$+ \int_0^\pi l(t)(1 + nt)^{-\gamma} O(nt)^{m+1} dt + O(1/n),$$

where $\gamma = \min(\beta + m + 2)$. Hence, there is a constant K such that

$$| r_n{}^\beta / n | \leqq K \sum_{k=0}^{m+1} n^{k-1} \int_0^{1/n} t^{k-1} \,|\, t l(t) \,|\, dt + K n^{-2} \int_{1/n}^{\pi} t^{-2} \,|\, t l(t) \,|\, dt$$

$$+ K n^{m-\gamma} \int_{1/n}^{\pi} t^{m-\gamma} \,|\, t l(t) \,|\, dt + O(1/n^2).$$

From this, the convergence of $\Sigma n^{-1} \,|\, r_n{}^\beta \,|$ follows, in view of Lemma 3.

Theorem 3 can also be deduced from Bosanquet's criterion. Indeed, we have the following theorem:

THEOREM 4. *If the repeated Cauchy integral* $l(t)$ *in Theorem 3 exists as a function integrable in the Lebesgue sense on* $(0, \pi)$, *then the mean function* $[\phi(t)]_{m+1}$ *of order* $1 + m$ *is of bounded variation in* $(0, \pi)$.

Write $\phi(t) = \phi_0(t) \equiv \phi_0$, and for $\nu > 0$,

$$\phi_\nu \equiv \phi_\nu(t) = \int_{+0}^{t} \phi_{\nu-1}(t)\, t^{-1} dt;$$

then $l(t) = t^{-1} \phi_m(t)$. From

$$t^{m+1}[\phi(t)]_{m+1}/(m + 1) = (\int_0^t dt)^{m+1} \phi(t) = (\int_0^t dt)^m \{t\phi_1 - \int_0^t \phi_1 dt\}$$

$$= (\int_0^t dt)^m t\phi_1 - (\int_0^t dt)^{m+1} \phi_1,$$

we infer that

$$t^{m+1}[\phi(t)]_{m+1}/(m+1)=t(\int_0^t dt)^m\phi_1-(m+1)(\int_0^t dt)^{m+1}\phi_1.$$

so that $t^{m+1}[\phi(t)]_{m+1}$ is a linear combination of the functions

$$t^m(\int_0^t dt)^1\phi_m,\ t^{m-1}(\int_0^t dt)^2\phi_m,\cdots,(\int_0^t dt)^{m+1}\phi_m.$$

Consequently, the function $[\phi(t)]_{m+1}$ is a linear combination of the functions

$$t^{-1}\int_0^t tl(t)\,dt,\ t^{-2}(\int_0^t dt)^2 tl(t),\cdots,t^{-1-m}(\int_0^t dt)^{1+m}tl(t);$$

or of the functions

$$th_1(t), th_2(t),\cdots, th_{m+1}(t).$$

On differentiation, we find that

$$(d/dt)[\phi(t)]_{m+1}=c_0 l(t)+c_1 h_1(t)+\cdots+c_{m+1}h_{m+1}(t),$$

where $c_0, c_1, \cdots, c_{m+1}$ denote constants. Since $l(t)$, and by Lemma 6, $h_\nu(t)$ are integrable in the Lebesgue sense on $(0,\pi)$, it follows that $[\phi(t)]'_{m+1}$ is absolutely integrable on $(0,\pi)$. Hence $[\phi(t)]_{m+1}$ is of bounded variation in $(0,\pi)$.

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THE LAGRANGE MULTIPLIER RULE FOR TWO DEPENDENT AND TWO INDEPENDENT VARIABLES.*

By Charles B. Barker, Jr.

Introduction. The Lagrange Multiplier Rule in the Calculus of Variations may be briefly described as follows. Suppose that $\bar{z}_1(x,y)$ and $\bar{z}_2(x,y)$ are functions which minimize an integral of the form

$$\int\!\!\int_G f(x, y, z_1, z_2, z_{1x}, z_{1y}, z_{2x}, z_{2y})\,dydx$$

among all functions $z_1(x,y)$ and $z_2(x,y)$ which assume certain prescribed boundary values on G^* and satisfy a given differential equation,

$$\phi(x, y, z_1, z_2, z_{1x}, z_{1y}, z_{2x}, z_{2y}) = 0.$$

Then there exists a function $\lambda(x,y)$, which is continuous together with its first derivative on the closed region $\bar{G}$ and is such that it satisfies the equations of variation,

$$(\partial/\partial x)(f_{z_{1x}} + \lambda\phi_{z_{1x}}) + (\partial/\partial y)(f_{z_{1y}} + \lambda\phi_{z_{1y}}) = f_{z_1} + \lambda\phi_{z_1}$$
$$(\partial/\partial x)(f_{z_{2x}} + \lambda\phi_{z_{2x}}) + (\partial/\partial y)(f_{z_{2y}} + \lambda\phi_{z_{2y}}) = f_{z_2} + \lambda\phi_{z_2}$$

in which the unwritten arguments of the partial derivatives of f and ϕ are $\bar{z}_1(x,y)$ and $\bar{z}_2(x,y)$ and their partial derivatives.

The existence of the function $\lambda(x,y)$ is demonstrated in this paper for all $\bar{z}_1(x,y)$ and $\bar{z}_2(x,y)$ which form a 'quasi-normal surface,' an explicit definition of which will be given later. The first result obtained in a study of the Lagrange problem for double integrals was that of Gross [4].[1] His result was re-established by Coral [3] under weakened conditions. A further result obtained by Coral in the same paper is closely related to the result of this paper. However, the region of integration considered by Coral is considerably more restricted than the one considered here, and Coral's differential equation does not contain all of the partial derivatives of the functions. On the other hand, the condition of 'quasi-normality' imposed upon the surfaces considered in the present paper appears to be more restrictive than the

* Received July 11, 1944.

[1] Numbers in brackets refer to the bibliography.

'normal' condition of Coral's. A very comprehensive treatment of the problem in one dimension, together with an extensive bibliography may be found in [2].

1. Notation. The following notation is employed. If G is a region, then $\bar{G}$ indicates its closure and G^* its boundary. A function, $f(x, y)$, is said to be of class $C^{(n)}$ on $\bar{G}$ if the function, together with its partial derivatives of the first n orders, are continuous on G and coincide with functions which are continuous on $\bar{G}$. A region G is said to be bounded by a simple, closed, regular curve of class $C_a^{(n)}$ if G^* is rectifiable and if the parametric equations of G^*, $x = x(s)$, $y = y(s)$, s being the arc length, are of class $C^{(n)}$ with $x^{(n)}(s)$ and $y^{(n)}(s)$ satisfying a uniform Hölder condition with exponent α, $0 < \alpha < 1$. The symbols p_i and q_i are used to represent, respectively,

$$(\partial/\partial x)[z_i(x, y)] \quad \text{and} \quad (\partial/\partial y)[z_i(x, y)], \qquad (i = 1, 2).$$

2. A preliminary transformation. In this section, a particular one-to-one transformation of the entire (x, y)-plane into the entire (ξ, η)-plane is established. This mapping possesses specific properties which are essential to the solution of the problem. In order to obtain the desired transformation, it is necessary to prove certain preliminary lemmas.

LEMMA 1. *Let G be a region bounded by a simple, closed, regular curve of class C_a'''. Suppose that $f(x, y)$ and $g(x, y)$ are functions of class C''' on $\bar{G}$, and that*

$$[f(x, y)]^2 + [g(x, y)]^2 > 0 \tag{1}$$

at each point of $\bar{G}$. Then there exist two functions, $r(x, y)$ and $\theta(x, y)$ which are of class C''' on $\bar{G}$ and which satisfy

$$r(x, y) \cos \theta(x, y) = f(x, y), \tag{2}$$

$$r(x, y) \sin \theta(x, y) = g(x, y) \tag{3}$$

at each point of $\bar{G}$.

Proof. Let

$$r(x, y) = + \{[f(x, y)]^2 + [g(x, y)]^2\}^{\frac{1}{2}}, \tag{4}$$

and let

$$\theta(x, y) = \int_{(x_0, y_0)}^{(x, y)} [A(x, y)\,dx + B(x, y)\,dy] + \theta(x_0, y_0) \tag{5}$$

where (x_0, y_0) is a point of $\bar{G}$, and where

$$A(x,y) = \frac{fg_x - f_x g}{f^2 + g^2}, \qquad B(x,y) = \frac{fg_y - f_y g}{f^2 + g^2}. \tag{6}$$

By Green's Theorem, if R is any rectangle contained in $\bar{G}$,

$$\int_{R^*} [A(x,y)dx + B(x,y)dy] = 0.$$

Hence, by a well known theorem (see, for example, [6]), $\theta(x,y)$ is of class C' on $\bar{G}$, and is such that

$$\partial\theta/\partial x = A(x,y), \qquad \partial\theta/\partial y = B(x,y). \tag{7}$$

Further, since

$$A(x,y) = (\partial/\partial x)(\text{arc tan } g/f), \qquad B(x,y) = (\partial/\partial y)(\text{arc tan } g/f),$$

it follows that

$$\cos\theta = f/\pm \{f^2 + g^2\}^{\frac{1}{2}}, \qquad \sin\theta = g/\pm \{f^2 + g^2\}^{\frac{1}{2}}, \tag{8}$$

where the signs in the denominators are to be taken the same. Thus, if $\theta(x_0, y_0)$ in (5) is chosen so that the positive sign holds in the first place, that sign will always hold on $\bar{G}$, since $\theta(x,y)$ is continuous there. In view of (1), (4), (6), (7), and (8), $r(x,y)$ and $\theta(x,y)$ are of class C''' on $\bar{G}$, and they clearly satisfy (2) and (3).

LEMMA 2. *Let G be a region bounded by a simple, closed, regular curve of class C_a''', and let $a(x,y)$, $b(x,y)$, $c(x,y)$, and $d(x,y)$ be functions of class C''' on $\bar{G}$, with*

$$[a(x,y)][d(x,y)] - [b(x,y)][c(x,y)] \neq 0 \tag{9}$$

at each point of $\bar{G}$. Then there exist four functions, $r_1(x,y)$, $\theta_1(x,y)$, $r_2(x,y)$, and $\theta_2(x,y)$ which are of class C''' on $\bar{G}$, and are such that

$$\begin{aligned} r_1(x,y)\cos\theta_1(x,y) &= a(x,y); \quad r_1(x,y)\sin\theta_1(x,y) = c(x,y); \\ r_2(x,y)\cos\theta_2(x,y) &= b(x,y); \quad r_2(x,y)\sin\theta_2(x,y) = d(x,y), \end{aligned} \tag{10}$$

on $\bar{G}$. In addition, $\theta_1(x,y)$ and $\theta_2(x,y)$ satisfy the inequality

$$0 < m \leq |\theta_2(x,y) - \theta_1(x,y)| \leq M < \pi, \tag{11}$$

at each point of $\bar{G}$.

Proof. Statement (9) implies that

$$[a(x,y)]^2 + [c(x,y)]^2 \neq 0, \quad \text{and} \quad [b(x,y)]^2 + [d(x,y)]^2 \neq 0$$

on $\bar{G}$. Thus Lemma 1 applies, and hence there exist four functions $\bar{r}_1(x,y)$, $\bar{\theta}_1(x,y)$, $\bar{r}_2(x,y)$, and $\bar{\theta}_2(x,y)$ which satisfy (10) on $\bar{G}$ and are of class C''' there. Now,

$$ad - bc = \bar{r}_1\bar{r}_2 \sin(\bar{\theta}_1 - \bar{\theta}_2),$$

and so, as a result of (9),

$$\bar{\theta}_2(x,y) - \bar{\theta}_1(x,y) \neq m\pi$$

for all integral values of m. Since both $\bar{\theta}_1(x,y)$ and $\bar{\theta}_2(x,y)$ are continuous, there exists an integer N such that

$$N\pi < [\bar{\theta}_2(x,y) - \bar{\theta}_1(x,y)] < (N+1)\pi,$$

or,

$$0 < [\{\bar{\theta}_2(x,y) - N\pi\} - \bar{\theta}_1(x,y)] < \pi.$$

Accordingly, if

$$\begin{aligned} r_1(x,y) &= \bar{r}_1(x,y), & \theta_1(x,y) &= \bar{\theta}_1(x,y), \\ r_2(x,y) &= \pm\, \bar{r}_2(x,y), & \theta_2(x,y) &= \bar{\theta}_2(x,y) - N\pi, \end{aligned}$$

where the plus or minus sign is taken according as N is even or odd, it is seen that equations (10) are satisfied and that

$$0 < |\,\theta_2(x,y) - \theta_1(x,y)\,| < \pi. \tag{12}$$

In view of the continuity of $\theta_1(x,y)$ and $\theta_2(x,y)$, (12) is equivalent to (11), thus concluding the proof of the lemma.

LEMMA 3. *Let $u_1(x)$, $u_2(x)$, $u_3(x)$, and $u_4(x)$ be functions of class C''', C'', C', and C, respectively. Then there exists a function $U(x,y)$ which is of class C''' for $y \geq 0$, and is such that*

$$\begin{aligned} U(x,0) &= u_1(x), & U_{yy}(x,0) &= u_3(x), \\ U_x(x,0) &= u_1'(x), & U_{xxx}(x,0) &= u_1'''(x), \\ U_y(x,0) &= u_2(x), & U_{xxy}(x,0) &= u_2''(x), \\ U_{xx}(x,0) &= u_1''(x), & U_{xyy}(x,0) &= u_3'(x), \\ U_{xy}(x,0) &= u_2'(x), & U_{yyy}(x,0) &= u_4(x). \end{aligned}$$

Proof. Let

$$v_1(x,y)=u_1(x);\qquad v_2(x,y)=(y/1!)[(1/2y)\int_{x-y}^{x+y}u_2(\xi)d\xi];$$

$$v_3(x,y)=(y^2/2!)\{(1/2y)\int_{x-y}^{x+y}[(1/2y)\int_{\xi-y}^{\xi+y}u_3(\eta)d\eta]d\xi\};$$

$$v_4(x,y)=y^3/3!\left\{(1/2y)\times\right.$$

$$\left.\int_{x-y}^{x+y}\{(1/2y)\int_{\xi-y}^{\xi+y}[(1/2y)\int_{\eta-y}^{\eta+y}(u_4(\sigma)-u_2''(\sigma))d\sigma]d\eta\}d\xi\right\}.$$

If, now, $U(x,y)$ be defined by

$$U(x,y)=v_1(x,y)+v_2(x,y)+v_3(x,y)+v_4(x,y),$$

it may easily be verified that $U(x,y)$ satisfies the conditions of the lemma.

LEMMA 4. *Let G be a region bounded by a simple, closed, regular curve of class C_a''', and let $a(x,y)$, $b(x,y)$, $c(x,y)$, $d(x,y)$, $e(x,y)$, and $f(x,y)$ be functions of class C''' on $\bar{G}$ with*

$$[a(x,y)][d(x,y)]-[b(x,y)][c(x,y)]\neq 0 \tag{13}$$

at each point of $\bar{G}$. Then these functions may be extended to be of class C''' over the entire plane, with all functions and their first three derivatives uniformly bounded and with (13) *satisfied over the entire plane.*

Proof. By Lemma 2, there exist four functions, $r_1(x,y)$, $\theta_1(x,y)$, $r_2(x,y)$, and $\theta_2(x,y)$ which satisfy (10) and (11). As a matter of convenience, let

$$\theta(x,y)=\theta_2(x,y)-\theta_1(x,y). \tag{14}$$

Now let

$$x=x(\xi,\eta),\qquad y=y(\xi,\eta) \tag{15}$$

be the conformal transformation of the exterior of G onto the exterior of Γ, the unit circle in the (ξ,η)-plane, the transformation being of class C''' both ways, even on G^* and Γ^*. The existence of such a transformation is well known [5]. Now, let $u(x,y)$ designate, in turn, each of the functions r_1, r_2, θ_1, θ_2, θ, e, and f, and let $u(\xi,\eta)$ be the transformed function on Γ^*. Define, on Γ^*,

$$u_\xi=u_xx_\xi+u_yy_\xi,\quad u_\eta=u_xx_\eta+u_yy_\eta,\quad u_\rho(1,\phi)=u_\xi\cos\phi+u_\eta\sin\phi,$$

(ρ,ϕ) being the polar coordinates in the (ξ,η)-plane, with u_x and u_y the known values on G^*, and similar definitions being made for the higher derivatives of $u(\rho,\phi)$.

Now let $h(\rho)$ be defined as a function of class C''' which is such that

$$(16)\quad h(1)=h'(1)=h''(1)=h'''(1)=h'(2)=h''(2)=h'''(2)=0,$$

$$(17)\quad h(2)=1.$$

Let u represent the functions r_1, r_2, $(2/\pi)\theta$, $(2/\pi)\theta_2$, θ_1, e, and f in turn. Then for the closed annulus, $1\leq\rho\leq 2$, let $u(\rho,\phi)$ be defined by

$$u(\rho,\phi)=U(\rho,\phi)[1-h(\rho)]+Ch(\rho),$$

where the constant C is unity for r_1, $(2/\pi)\theta$, $(2/\pi)\theta_2$, e, and f, is zero for θ_1, and is ± 1 for r_2, the sign being that of r_2 on Γ^*, and where $U(\rho,\phi)$ is the function $U(x,y)$ of Lemma 3, defined for

$$u_1(x)=u(1,\phi),\qquad u_3(x)=u_{\rho\rho}(1,\phi),$$
$$u_2(x)=u_\rho(1,\phi),\qquad u_4(x)=u_{\rho\rho\rho}(1,\phi).$$

Clearly $u(\rho,\phi)$ is of class C''' for this region, and it may easily be verified that $u(\rho,\phi)$ matches up, together with its first three derivatives, with the known values on Γ^*.

From (4), (11), and (14), it is observed that r_1, r_2, and θ possess bounds on the unit circle of the form

$$|r_1|\geq k_1>0,\quad |r_2|\geq k_2>0,\quad 0<2m/\pi\leq|(2/\pi)\theta|\leq 2M/\pi<2.$$

These bounds cannot necessarily be preserved on the annulus, but it is possible to maintain the following set of conditions there:

$$(18)\quad |r_1|\geq k_1/2>0,\ |r_2|\geq k_2/2>0,\ 0<m/\pi\leq|(2/\pi)\theta|\leq(M+\pi)/\pi<2.$$

Suppose that ρ_1 is the smallest value of ρ at which any violation of (18) occurs. Then let $h(\rho)$ satisfy (16), be monotone, and satisfy

$$h(\rho)=1\ \text{ for }\ (1+\rho_1)/2\leq\rho=2.$$

Clearly this preserves the bounds of (18). The other functions, together with all derivatives, are evidently bounded on the closed annulus, since each is continuous on a closed set.

For $\rho>2$, let $u(\rho,\phi)$ and its first three derivatives be defined to possess their respective values at $\rho=2$. As these values are all constants, $u(\rho,\phi)$ is clearly of class C''' for $\rho>2$, and all functions, together with their derivatives, are bounded. In particular,

$$(2/\pi)\theta(2,\phi)=r_1(2,\phi)=|r_2(2,\phi)|=1,$$

so that (13) is seen to be satisfied for all $\rho>2$.

Now let the inverse transformation of (15) be applied. The functions all return to their original values on G^* and all derivatives match up on G^*. Outside of G^*, the functions are of class C''' and are uniformly bounded, as are all derivatives. In particular, $(2/\pi)\theta$, r_1, and r_2 are bounded by the transformed conditions (18), and thus (13) is satisfied everywhere.

As an additional result of this lemma, due to the nature of the boundary of G, there exists a square having G^* in its interior, and having the functions θ_1 and θ_2 equal to 0 and $\pi/2$, respectively, on the boundary and outside of the square.

LEMMA 5. *Let the curve C,*

$$x = x(s), \quad y = y(s), \quad a \leq s \leq a + k,$$

s being the arc length, be a simple closed curve with no cusps, and be such that the interval $(a, a + k)$ may be divided into a finite number of subintervals, (s_{i-1}, s_i) in each of which closed intervals $x(s)$ and $y(s)$ are of class C'''. Let $x(s)$ and $y(s)$ be defined for all values of s by:

$$x(s \pm k) = x(s), \quad y(s \pm k) = y(s).$$

Then there exists a function, $\theta(s)$, uniquely determined except possibly for a multiple of 2π, which is such that

$$\begin{gathered}
|\theta([a + k]^+) - \theta(a^+)| = 2\pi, \\
\cos\theta(s) = x'(s), \quad \sin\theta(s) = y'(s), \quad s \neq s_i, \\
\cos\theta(s_i^+) = x'(s_i^+), \quad \sin\theta(s_i^+) = y'(s_i^+), \\
\cos\theta(s_i^-) = x'(s_i^-), \quad \sin\theta(s_i^-) = y'(s_i^-), \\
|\theta(s_i^+) - \theta(s_i^-)| < \pi.
\end{gathered}$$

Proof. This is a well known theorem, and may be found in [7].

LEMMA 6. *Let $\theta(x, y)$ be of class C''' over the entire plane with $|\theta|$, $|\theta_x|$, and $|\theta_y|$ all bounded by M for all (x, y). Let $x = x(s)$, $y = y(s)$, s being the arc length, be the solution of*

$$\text{(19)} \qquad dx/ds = \cos\theta, \qquad dy/ds = \sin\theta,$$

for which

$$x(0) = x_0, \qquad y(0) = y_0.[2]$$

Then

[2] The existence and uniqueness of these functions is guaranteed by well known existence theorems in the theory of ordinary differential equations, and may be found in [1].

(20) $$x(s_1) = x(s_2), \qquad y(s_1) = y(s_2)$$

implies that

(21) $$s_1 = s_2.$$

Proof. If one assumes (20) to be satisfied and (21) to be not satisfied, a contradiction can be readily obtained through the use of Lemma 5.

LEMMA 7. *Let $\theta_1(x, y)$ and $\theta_2(x, y)$ be of class C' over the entire plane, be uniformly bounded together with their first derivatives there, and be such that*

$$0 < m \leq | \theta_2(x, y) - \theta_1(x, y) | \leq M < \pi$$

for all (x, y). Then there exist unique functions of class C''

$$x_1(s), \; y_1(s), \; x_2(t), \text{ and } y_2(t),$$

such that

(22A) $$dx_1/ds = \cos\theta_1, \qquad dy_1/ds = \sin\theta_1$$

(22B) $$dx_2/dt = \cos\theta_2, \qquad dy_2/dt = \sin\theta_2,$$

and such that

$$x_1(0) = x_1, \; y_1(0) = y_1, \; x_2(0) = x_1, \; y_2(0) = y_1.$$

For these functions,

$$x_1(s_1) = x_2(t_2), \text{ and } y_1(s_1) = y_2(t_2)$$

imply that

$$s_1 = t_2 = 0.$$

Proof. This lemma follows as a consequence of the two preceding lemmas.

LEMMA 8. *Let $\theta_1(x, y)$ and $\theta_2(x, y)$ be of class C''' over the entire plane with $|\theta_1|$, $|\theta_2|$, $|\theta_{1x}|$, $|\theta_{1y}|$, $|\theta_{2x}|$, and $|\theta_{2y}|$ all bounded by $\bar{M}$ for all (x, y). Suppose also that*

$$0 < m \leq | \theta_2(x, y) - \theta_1(x, y) | \leq M < \pi$$

for all (x, y). Let $x_1(s)$ and $y_1(s)$ satisfy (22A) of Lemma 7, and let $x_1(s, t)$ and $y_1(s, t)$ be defined for each s by

$$(\partial/\partial t)[x_1(s, t)] = \cos\theta_2, \qquad (\partial/\partial t)[y_1(s, t)] = \sin\theta_2$$

with

$$x_1(s, 0) = x_1(s), \qquad y_1(s, 0) = y_1(s).$$

Then $x_1(s, t)$ and $y_1(s, t)$ are of class C''' and the transformation

$$x = x_1(s, t), \qquad y = y_1(s, t)$$

is one-to-one from the entire (s, t)-plane into a subset of the (x, y)-plane.

Proof. This follows readily from Lemma 7.

LEMMA 9. *Let $\theta(x, y)$ be of class C''' over the entire plane with $|\theta|$, $|\theta_x|$, and $|\theta_y|$ all bounded by M for all (x, y). Let*

$$x = x(s), \qquad y = y(s),$$

s being the arc length, be the solutions of

$$dx/ds = \cos\theta, \qquad dy/ds = \sin\theta,$$

for which

$$x(0) = x_0, \qquad y(0) = y_0.$$

Then for each R there exists an $s_1 > 0$ and an $s_2 > 0$ such that

$$[x(s_1) - x_0]^2 + [y(s_1) - y_0]^2 \geq R^2, \tag{23}$$

$$[x(-s_2) - x_0]^2 + [y(-s_2) - y_0]^2 \geq R^2. \tag{24}$$

Proof. Let $x_1(s, t)$ and $y_1(s, t)$ be determined by

$$\partial x/\partial t = -\sin\theta, \qquad \partial y/\partial t = \cos\theta$$

with

$$x_1(s, 0) = x(s), \qquad y_1(s, 0) = y(s). \tag{25}$$

Then by Lemma 8, the transformation

$$x = x_1(s, t), \qquad y = y_1(s, t)$$

is one-to-one from the entire (s, t)-plane into a subset of the (x, y)-plane. Let

$$J_1(s, t) = x_{1s}y_{1t} - x_{1t}y_{1s},$$

so that

$$J_1(s, 0) = 1.$$

Also, since

$$J_{1t} = -(\theta_x \cos\theta + \theta_y \sin\theta) \cdot J_1,$$

it follows that

$$e^{-2M|t|} \leq J_1(s, t) \leq e^{2M|t|}. \tag{26}$$

In addition,

$$\{x_{1t}^2 + y_{1t}^2\}^{\frac{1}{2}} = 1.$$

Now, let

$$A(s) = \int_{-T}^{+T} \int_0^s J_1(s, t)\, ds dt.$$

Integrating this and making use of (26),

$$A(s) \geq (s/M)(1-e^{-2TM}),$$

and so

(27) $$\lim_{s\to\infty} A(s) = +\infty$$

In particular, if $T=1$,

$$A(s) \geq (s/M)(1-e^{-2M}).$$

Since (27) holds, there exists an s_1 such that the area in the (x,y)-plane corresponding to the set

$$0 \leq s \leq s_1, \qquad -1 \leq t \leq 1$$

exceeds $(R+1)^2$. That is, there exists a point (s_0, t_0) in this set such that

$$[x_1(s_0,t_0)-x_0]^2 + [y_1(s_0,t_0)-y_0]^2 > (R+1)^2.$$

In addition,

$$[x_1(s_0,t_0)-x_1(s_0,0)]^2 + [y_1(s_0,t_0)-y_1(s_0,0)]^2 \leq 1$$

so that

$$[x_1(s_0,0)-x_0]^2 + [y_1(s_0,0)-y_0]^2 > R^2.$$

Thus, in view of (25), it is evident that (23) holds. A similar method of attack will demonstrate the validity of the inequality (24).

LEMMA 10. *Let G be a region bounded by a simple, closed regular curve of class C_a''', and let $a(x,y)$, $b(x,y)$, $c(x,y)$, and $d(x,y)$ be functions of class C''' on $\bar{G}$, with*

$$[a(x,y)][d(x,y)] - [b(x,y)][c(x,y)] \neq 0$$

at each point of $\bar{G}$. Then there exists a one-to-one transformation

(28) $$\xi = \xi(x,y), \qquad \eta = \eta(x,y)$$

of the entire (x,y)-plane into the entire (ξ,η)-plane, which, together with its inverse, is of class C''' over the entire plane, and in which ξ and η satisfy

(29) $$\begin{aligned} \xi_x\eta_y - \xi_y\eta_x &\neq 0, \\ a\eta_x + c\eta_y &= 0, \\ b\xi_x + d\xi_y &= 0. \end{aligned}$$

at each point of $\bar{G}$.

Proof. Let the extension of Lemma 4 be applied to these four given functions. Then the functions $r_1(x,y)$, $\theta_1(x,y)$, $r_2(x,y)$ and $\theta_2(x,y)$, which are the functions of Lemma 2, are all of class C''' over the entire plane, and, together with their first three derivatives, are uniformly bounded everywhere.

In particular, there exists a square S: $[(x_0, y_0), (x_0 + k, y_0 + k)]$ which contains $\bar{G}$ in its interior, and which is such that $\theta_1(x, y)$ and $\theta_2(x, y)$ equal 0 and $\pi/2$, respectively, on S^* and at all points exterior to S^*, as well as at all points (x, y) interior to S which are such that their distance from S^* is not greater than unity.

Let

$$x = x_1(s), \; y = y_1(s), \text{ and } x = x_2(t), \; y = y_2(t)$$

be the families of curves satisfying (22A) and (22B), respectively, and designate by C_1 and C_2 the curves of the first and the second families, respectively, through (x_0, y_0). Now let P: (x, y) be any point, and let C'_1 and C'_2 be the curves of the first and second families through P. By means of Lemma 9, we can easily show that C'_1 intersects C_2, and that C'_2 intersects C_1. Let P_1 and P_2 be, respectively, these two points of intersection. Then let s be the directed distance along C_1 from P to P_1, and t be the directed distance along C_2 from P to P_2. Let $\xi(x, y)$ and $\eta(x, y)$ be defined by

$$\xi(x, y) = s, \qquad \eta(x, y) = t.$$

By Lemma 8 and the above, the transformation

$$\xi = \xi(x, y), \qquad \eta = \eta(x, y)$$

is one-to-one from the entire (x, y)-plane into the entire (ξ, η)-plane.

Let $x_1(s, t)$ and $y_1(s, t)$ be defined by

$$(\partial/\partial s)[x_1(s, t)] = \cos\theta_1(x_1, y_1), \qquad (\partial/\partial s)[y_1(s, t)] = \sin\theta_1(x_1, y_1)$$

with

$$x_1(0, t) = x_2(t), \qquad y_1(0, t) = y_2(t),$$

$x = x_2(t)$, $y = y_2(t)$ being the equations of C_2. Let $x_2(s, t)$ and $y_2(s, t)$ be defined by

$$(\partial/\partial t)[x_2(s, t)] = \cos\theta_2(x_2, y_2), \qquad (\partial/\partial t)[y_2(s, t)] = \sin\theta_2(x_2, y_2)$$

with

$$x_2(s, 0) = x_1(s), \qquad y_2(s, 0) = y_1(s),$$

$x = x_1(s)$, $y = y_1(s)$ being the equations of C_1.

By Lemma 8, all of the functions are of class C''' in both s and t. Also, it follows that both of the transformations,

$$x = x_i(s, t), \qquad y = y_i(s, t), \qquad (i = 1, 2),$$

carry the entire (s, t)-plane into the entire (x, y)-plane.

If these transformations are reversed so as to obtain

(30) $\quad s = s_1(x, y),\ t = t_1(x, y),$ and $s = s_2(x, y),\ t = t_2(x, y),$

from the first and second, respectively, then it is seen that

$$\xi = \xi(x, y) = s_2(x, y), \qquad \eta = \eta(x, y) = t_1(x, y).$$

Now, let

$$J_i(s, t) = x_{is}y_{it} - x_{it}y_{is}, \qquad (i = 1, 2);$$

it may easily be verified that $J_1(0, t)$ is never zero, and that since

$$J_{1s} = [(\partial/\partial y) \sin \theta_1 + (\partial/\partial x) \cos \theta_1] \cdot J_1,$$

$J_1(s, t)$ is never zero. Similarly, $J_2(s, t)$ is never zero. Thus the inverse transformations (30) are of class C''', and hence both ξ and η are of class C'''.

Now let

$$J(s, t) = \xi_x\eta_y - \xi_y\eta_x.$$

This may easily be reduced to

$$J = \sin [\theta_2(x, y) - \theta_1(x, y)]/J_1J_2.$$

Thus J is defined everywhere and never vanishes. Hence the inverse of (28) is of class C''' everywhere.

Now,

$$\begin{aligned} a\eta_x + c\eta_y &= r_1 \cos \theta_1\eta_x + r_1 \sin \theta_1\eta_y \\ &= r_1[x_{1s}\eta_x + y_{1s}\eta_y] \equiv 0, \end{aligned}$$

since J_1 never vanishes and since

$$\eta_x = - y_{1s}/J_1, \qquad \eta_y = x_{1s}/J_1.$$

Similarly, it may be shown that

$$b\xi_x + d\xi_y = 0.$$

This is the transformation sought for in this section. The essential features that will be needed later on are its differentiability properties, the three statements of (28), and the fact that it is one-to-one from the entire (x, y)-plane into the entire (ξ, η)-plane.

3. A lemma of general interest in the calculus of variations. In this section, we seek a lemma which is of quite general interest in the Calculus of Variations. It is necessary to develop several preliminary lemmas which yield, upon proper application of the transformation of the preceding section, the desired result.

LEMMA 11. *Let $B(x,y)$, $\alpha(x,y)$, and $\epsilon(x,y)$ be functions of class C'' on the closed rectangle $\bar{R}$: $[(x_1, y_1), (x_2, y_2)]$, with $B(x,y)$ and $\alpha(x,y)$ never zero on $\bar{R}$. Then there exists a function $u(x,y)$ which is of class C''' in x and of class C'' in y which possesses the boundary values*

$$u(x, y_1) = u(x, y_2) = u_y(x, y_1) = u_y(x, y_2) = 0,$$

and which is such that

$$\int_{y_1}^{y_2} B(x,y)[\alpha(x,y)u_x(x,y) + \epsilon(x,y)u(x,y)]dy \overset{x}{\equiv} 1.$$

Proof. Let $v(x,y)$ be of class C' and be such that

$$\alpha(x,y)v_x(x,y) + \epsilon(x,y)v(x,y) = 1/B(x,y).$$

If now, $\phi(y)$ is any function of class C'' satisfying

$$\phi(y_1) = \phi(y_2) = \phi'(y_1) = \phi'(y_2) = 0$$

and such that

$$\int_{y_1}^{y_2} \phi(t)\,dt \neq 0,$$

then obviously the required function of the lemma is

$$u(x,y) = v(x,y)\phi(y) / \int_{y_1}^{y_2} \phi(t)\,dt.$$

LEMMA 12. *Let $\phi(x,y)$ be a function of class C' on $\bar{R}$, $\bar{R}$ being the rectangle: $[(x_1, y_1), (x_2, y_2)]$, and suppose that*

$$\int_R\int \phi(x,y)u(x,y)\,dy\,dx = 0 \tag{31}$$

for all $u(x,y)$ which are of class C'', with $u_{xxx}(x,y)$ also continuous and zero on R^*, which vanish together with their first and second derivatives on R^*, and which satisfy

$$\int_{y_1}^{y_2} B(x,y)[\alpha(x,y)u_x(x,y) + \epsilon(x,y)u(x,y)]dy \overset{x}{\equiv} 0, \tag{32}$$

$B(x,y)$, $\alpha(x,y)$, and $\epsilon(x,y)$ satisfying the conditions stated for them in the preceding lemma.

Then there exists a function $\chi(x)$ of class C' for $x_1 \leq x \leq x_2$ such that

$$\phi(x,y) + (\partial/\partial x)[B(x,y)\alpha(x,y)\chi(x)] - B(x,y)\,\epsilon\,(x,y)\chi(x) = 0 \tag{33}$$

on $\bar{R}$.

Proof. Let $u_1(x,y)$ be the function of Lemma 11. Now let $v(x,y)$ be any function of class C''' which vanishes together with its derivatives on R^*, and define $\psi(x)$ by

$$\psi(x) = \int_{y_1}^{y_2} B(x,t)[\alpha(x,t)v_x(x,t) + \epsilon(x,t)v(x,t)]dt.$$

Thus $\psi(x)$ is of class C'', and satisfies

$$\psi(x_1) = \psi(x_2) = \psi'(x_1) = \psi'(x_2) = 0.$$

Now let

$$v(x,y) = u^*(x,y) + u_1(x,y)\psi(x).$$

Obviously $u^*(x,y)$ satisfies both (31) and (32). If, now,

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} v(x,y)\phi(x,y)\,dydx$$

be manipulated by various interchanges of letters and orders of integration, as well as by integration by parts, there results the equation,

$$\begin{aligned}(34)\qquad \int_{x_1}^{x_2}\int_{y_1}^{y_2} v(x,y)\{\phi(x,y) - B(x,y)\,\epsilon\,(x,y)\chi(x)& \\ + (\partial/\partial x)[B(x,y)\alpha(x,y)\chi(x)]\}dydx &= 0.\end{aligned}$$

where

$$\chi(x) = \int_{y_1}^{y_2} \phi(x,t)u_1(x,t)dt.$$

Since (34) is satisfied by all $v(x,y)$ of class C'' which vanish together with their first and second derivatives on R^*, then the statement (33) follows immediately from the fundamental lemma of the calculus of variations.

LEMMA 13. *Let $A(x,y)$, $B(x,y)$, $C(x,y)$, $D(x,y)$, $E(x,y)$, and $F(x,y)$ be functions of class C'' on a simply connected region Γ with $AD - BC \neq 0$ on Γ. Then a necessary and sufficient condition that there exists a solution $\mu \not\equiv 0$ of class C' on Γ of the two equations*

$$(35)\quad (\partial/\partial x)(A\mu) + (\partial/\partial y)(C\mu) = E\mu, \quad \textit{and} \quad (\partial/\partial x)(B\mu) + (\partial/\partial y)(D\mu) = F\mu$$

is that

$$(36)\qquad \frac{\partial}{\partial y}\left[\frac{D(E - A_x - C_y) - C(F - B_x - D_y)}{AD - BC}\right] = \frac{\partial}{\partial x}\left[\frac{A(F - B_x - D_y) - B(E - A_x - C_y)}{AD - BC}\right]$$

on Γ.

Proof. If the equations (35) be solved simultaneously for μ_x and μ_y, it is found that

$$\mu_x = \frac{D(E - A_x - C_y) - C(F - B_x - D_y)}{AD - BC} \cdot \mu,$$

$$\mu_y = \frac{A(F - B_x - D_y) - B(E - A_x - C_y)}{AD - BC} \cdot \mu,$$

From these, we may conclude that if $\mu \not\equiv 0$ on Γ, then μ is never zero on Γ, and so $\log \mu$ is defined and is of class C'' on Γ. Dividing these last two equations by μ, it is clear that, if (35) holds, then (36) is also satisfied.

Finally, if (36) holds, there exists a function $\phi(x, y)$ of class C'' on Γ such that

$$\phi_x = \frac{D(E - A_x - C_y) - C(F - B_x - D_y)}{AD - BC},$$

$$\phi_y = \frac{A(F - B_x - D_y) - B(E - A_x - C_y)}{AD - BC},$$

since Γ is simply connected. Thus any function μ of the form

$$\mu = Ce^{\phi},$$

C being any constant, satisfies (35).

LEMMA 14. *Let* $a(x, y)$, $b(x, y)$, $c(x, y)$, $d(x, y)$, $e(x, y)$, $f(x, y)$, $\alpha(x, y)$, $\delta(x, y)$, $\epsilon(x, y)$, *and* $\zeta(x, y)$ *be of class* C'' *on the rectangle* R: $[(x_1, y_1), (x_2, y_2)]$ *with* $\alpha \cdot \delta \neq 0$ *on* $\bar{R}$. *Let* $u(x, y)$ *and* $v(x, y)$ *be of class* C' *on* $\bar{R}$, *vanish on* R^*, *and satisfy*

$$\alpha u_x + \delta v_y + \epsilon u + \zeta v = 0 \tag{37}$$

on $\bar{R}$.

Then

$$\int_R\int (au_x + bv_x + cu_y + dv_y + eu + fv)\,dydx \tag{38}$$
$$= \int_R\int u[(\partial/\partial x)(\alpha H) - \epsilon H - (a_x + c_y - e)]\,dydx,$$

where

$$H(x, y) = -\frac{1}{\delta(x, y)} \exp\left[\int_{y_1}^{y} \frac{\zeta(x, t)}{\delta(x, t)}\,dt\right]$$
$$\times \int_y^{y_2} \exp\left[-\int_{y_1}^{t} \frac{\zeta(x, r)}{\delta(x, r)}\,dr\right] \cdot [b_x(x, t) + d_y(x, t) - f(x, t)]\,dt.$$

Proof. If $u(x, y)$ and $v(x, y)$ satisfy (37) on $\bar{R}$ and vanish on R^*, we have

$$v(x, y) = K(x, y) \int_{y_1}^{y} B(x, t)[\alpha(x, t)u_x(x, t) + \epsilon(x, t)u(x, t)]\,dt, \tag{39}$$

where

$$K(x,y) = -\exp\left[-\int_{y_1}^{y} \frac{\zeta(x,r)}{\delta(x,r)}\,dr\right],$$

and

$$B(x,y) = \frac{1}{\delta(x,y)} \exp\left[\int_{y_1}^{y} \frac{\zeta(x,r)}{\delta(x,r)}\,dr\right].$$

Now if the left hand member of (38) be integrated by parts, and $v(x,y)$ be replaced by its equal from (39), there will result, after various interchanges of the order of integration as well as several integrations by parts, the right hand side of (38), as required.

LEMMA 15. *Let $a(x,y)$, $b(x,y)$, $c(x,y)$, $d(x,y)$, $e(x,y)$, $f(x,y)$, $\alpha(x,y)$, $\delta(x,y)$, $\epsilon(x,y)$, and $\zeta(x,y)$ be of class C'' with $\alpha\cdot\delta \neq 0$ on the closed rectangle $\bar{R}$: $[(x_1,y_1),(x_2,y_2)]$. Suppose that*

$$\int_R\int (au_x + bv_x + cu_y + dv_y + eu + fv)\,dydx = 0$$

for all pairs of functions $u(x,y)$ and $v(x,y)$ which are of class C'' on $\bar{R}$, which vanish with their first and second derivatives on R^*, and which satisfy

$$\alpha u_x + \delta v_y + \epsilon u + \zeta v = 0 \tag{40}$$

on $\bar{R}$.

Then there exists at least one function $\lambda(x,y)$ of class C' on $\bar{R}$ which satisfies the two equations

$$(\partial/\partial x)(a - \lambda\alpha) + (\partial/\partial y)(c) = e - \lambda\epsilon;$$
$$(\partial/\partial x)(b) + (\partial/\partial y)(d - \lambda\delta) = f - \lambda\zeta$$

on $\bar{R}$.

Proof. Let $u(x,y)$ and $v(x,y)$ be of class C'' on $\bar{R}$, vanish with their first and second derivatives on R^*, and satisfy (40). Suppose that $u_{xxx}(x,y)$ is also continuous and vanishes on R^*. Then the equation (39) of Lemma 14 holds. In order for $v(x,y)$ to vanish on R^*, it must be true that

$$\int_{y_1}^{y_2} B(x,t)[\alpha(x,t)u_x(x,t) + \epsilon(x,t)u(x,t)]\,dt \overset{x}{\equiv} 0. \tag{41}$$

Thus, since all of the conditions of Lemma 14 are satisfied, the result stated as equation (38) must hold for all $u(x,y)$ satisfying all the stated conditions, including (41). From Lemma 12, then, it follows that there exists a function $\chi(x)$ of class C', such that

$$(\partial/\partial x)(\alpha H) - \epsilon H - (a_x + c_y - e) + (\partial/\partial x)(\alpha B\chi) - \epsilon B\chi = 0,$$

where H and B are defined as in Lemma 14.

Now, if we define

$$\lambda(x,y) = H(x,y) + B(x,y)\chi(x),$$

it may easily be verified that this function satisfies the required conditions of the lemma.

LEMMA 16. *Let G be a region bounded by a simple, closed, regular curve of class C_a'''. Let $A(x,y)$, $B(x,y)$, $C(x,y)$, $D(x,y)$, $E(x,y)$, $F(x,y)$, $a(x,y)$, $b(x,y)$, $c(x,y)$, $d(x,y)$, $e(x,y)$, and $f(x,y)$ be functions of class C''' on $\bar{G}$, with*

$$[A(x,y)][D(x,y)] - [B(x,y)][C(x,y)] \neq 0$$

on $\bar{G}$. Suppose also that the set of points (x,y) where

$$\frac{\partial}{\partial y}\left[\frac{D(E-A_x-C_y)-C(F-B_x-D_y)}{AD-BC}\right] = \frac{\partial}{\partial x}\left[\frac{A(F-B_x-D_y)-B(E-A_x-C_y)}{AD-BC}\right] \tag{42}$$

is nowhere dense. Suppose finally that

$$\int_R\int (au_x + bv_x + cu_y + dv_y + eu + fv)\,dydx = 0 \tag{43}$$

for all $u(x,y)$ and $v(x,y)$ which are of class C'' on $\bar{G}$, which vanish together with their first and second derivatives on G^*, and which satisfy

$$Au_x + Bv_x + Cu_y + Dv_y + Eu + Fv = 0. \tag{44}$$

Then there exists a unique function $\lambda(x,y)$, which is of class C' on $\bar{G}$, and which is such that

$$\begin{aligned}(\partial/\partial x)(a-\lambda A) + (\partial/\partial y)(c-\lambda C) &= e-\lambda E,\\ (\partial/\partial x)(b-\lambda B) + (\partial/\partial y)(d-\lambda D) &= f-\lambda F.\end{aligned} \tag{45}$$

Proof. Let the transformation of Lemma 10 be applied to these given functions. Then (43) becomes, Γ being the transform of G,

$$\int_\Gamma\int (a_1u_\xi + b_1v_\xi + c_1u_\eta + d_1v_\eta + e_1u + f_1v)\,d\eta d\xi = 0 \tag{46}$$

where

$$a_1(\xi,\eta) = \{a[x(\xi,\eta),y(\xi,\eta)]\xi_x + c[x(\xi,\eta),y(\xi,\eta)]\xi_y\}(x_\xi y_\eta - x_\eta y_\xi),$$

with b_1, c_1, d_1, e_1, and f_1 represented by similar expressions, and where $u(\xi,\eta)$ and $v(\xi,\eta)$ are the transforms of $u(x,y)$ and $v(x,y)$. Equation (44) becomes

$$\begin{aligned}(A_1\xi_x + C_1\xi_y)u_\xi + (B_1\xi_x + D_1\xi_y)v_\xi + (A_1\eta_x + C_1\eta_y)u_\eta&\\ + (B_1\eta_x + D_1\eta_y)v_\eta + E_1u + F_1v = 0,&\end{aligned} \tag{47}$$

A_1, B_1, C_1, D_1, E_1, and F_1 being the transformed functions A, B, C, D, E, and F respectively. From the properties of the transformation, the coefficients of u_η and v_ξ are zero, and, in addition,

$$(A_1\xi_x + C_1\xi_y)(B_1\eta_x + D_1\eta_y) = (A_1D_1 - B_1C_1)(\xi_x\eta_y - \xi_y\eta_x) \neq 0,$$

so that (47) may be written as

$$\alpha u_\xi + \delta v_\eta + \epsilon u + \zeta v = 0,$$

α, δ, ϵ, and ζ having the obvious interpretations, with

$$\alpha(\xi, \eta)\delta(\xi, \eta) \neq 0.$$

Also $u(\xi, \eta)$ and $v(\xi, \eta)$ are of class C'' and vanish with their first two derivatives on Γ^*.

Now let $\bar{\rho}$ be a rectangle in $\bar{\Gamma}$, and let $\bar{R}$ be the corresponding region in $\bar{G}$. Then, by Lemma 15, there exists a function $\lambda_1(\xi, \eta)$ which is of class C' on $\bar{\rho}$ and which satisfies

$$(48) \qquad \begin{aligned} (\partial/\partial\xi)(a_1 - \lambda_1\alpha) + (\partial/\partial\eta)(c_1) &= e_1 - \lambda_1\epsilon, \\ (\partial/\partial\xi)(b_1) + (\partial/\partial\eta)(d_1 - \lambda_1\delta) &= f_1 - \lambda_1\zeta. \end{aligned}$$

Now let a, A, etc., be the original functions of x and y with $x(\xi, \eta)$ and $y(\xi, \eta)$ substituted in when the context demands. Let

$$J = \xi_x\eta_y - \xi_y\eta_x, \qquad J_1 = x_\xi y_\eta - x_\eta y_\xi.$$

On rewriting the first of equations (48), using the formulas for a_1, α, etc., we have

$$\frac{\partial}{\partial\xi}\left[\frac{a\xi_x + c\xi_y}{J} - \lambda_1(A\xi_x + C\xi_y)\right] + \frac{\partial}{\partial\eta}\left[\frac{a\eta_x + c\eta_y}{J}\right] = e/J - \lambda_1 E.$$

Now by manipulation of this last expression, including the use of formulas for the derivatives of the inverse functions, as well as certain properties of the transformation, we obtain

$$a_x + c_y - \lambda(A_x + C_y) - A\lambda_x - C\lambda_y = e - \lambda E,$$

where

$$\lambda = \lambda_1/J_1.$$

This is the first of the equations of (45), and the second may be obtained analogously.

Thus on each region $\bar{R}$ of the type indicated, there exists at least one $\lambda(x, y)$ satisfying (45). Since the set where (42) holds is nowhere dense

in $\bar{G}$, it follows that (42) cannot hold identically on $\bar{R}$, and it follows from Lemma 13 that $\lambda(x,y)$ is uniquely determined on $\bar{R}$ and hence over the interior of G.

Finally, if $\lambda_1(\xi,\eta) = \lambda(x,y)/J$, it can easily be shown that λ_1, $\lambda_{1\xi}$, and $\lambda_{1\eta}$ have continuous limits on Γ^* by studying the behavior of these functions along a small part of Γ^*.

4. The Lagrange multiplier rule. In this final section, we wish to prove the Lagrange Multiplier Rule for the case of two dependent variables and two independent variables.

Suppose, then, that $f(x,y,z_1,z_2,p_1,p_2,q_1,q_2)$ and $\phi(x,y,z_1,z_2,p_1,p_2,q_1,q_2)$ are of class C'''' in their arguments, that $\bar{z}_1(x,y)$ and $\bar{z}_2(x,y)$ are of class C'''' on a region $\bar{G}$, the boundary of which is a simple, closed, regular curve of class C_a''', and that $\bar{z}_1(x,y)$ and $\bar{z}_2(x,y)$ minimize

$$(49) \qquad \int_G\int f(x,y,z_1,z_2,p_1,p_2,q_1,q_2)\,dydx$$

among all functions z_1, z_2 of class C'' on $\bar{G}$, coinciding with $\bar{z}_1$ and $\bar{z}_2$, respectively, on G^*, and satisfying

$$(50) \qquad \phi(x,y,z_1,z_2,p_1,p_2,q_1,q_2) = 0.$$

Now suppose for the moment that $Z_1(x,y;\mu)$ and $Z_2(x,y;\mu)$ are of class C'' in their arguments for (x,y) on $\bar{G}$ and $|\mu| < \mu_0$, coincide with $\bar{z}_1$ and $\bar{z}_2$ if $\mu = 0$, and satisfy (50) for $|\mu| < \mu_0$. Then, if we differentiate with respect to μ and then set $\mu = 0$, we see that

$$(51) \qquad \bar{\phi}_{p_1}\zeta_{1x} + \bar{\phi}_{p_2}\zeta_{2x} + \bar{\phi}_{q_1}\zeta_{1y} + \bar{\phi}_{q_2}\zeta_{2y} + \bar{\phi}_{z_1}\zeta_1 + \bar{\phi}_{z_2}\zeta_2 = 0$$

and ζ_1 and ζ_2 vanish on G^*. Here we have set

$$\zeta_1 = [(\partial/\partial\mu)Z_1(x,y;\mu)]_{\mu=0}, \qquad \zeta_2 = [(\partial/\partial\mu)Z_2(x,y;\mu)]_{\mu=0}$$

and $\bar{\phi}_{p_1}$, etc., stand for $\phi_{p_1}(x,y,\bar{z}_1,\bar{z}_2,\bar{z}_{1x},\bar{z}_{2x},\bar{z}_{1y},\bar{z}_{2y})$, etc. Equation (51) is known as the equation of variation.

We cannot prove the multiplier rule for all possible cases, but shall prove it only for the case where the surface $z_1 = \bar{z}_1(x,y)$, $z_2 = \bar{z}_2(x,y)$ is 'quasi-normal' with respect to the differential equation (50).

Definition. The surface $z_1 = \bar{z}_1(x,y)$, $z_2 = \bar{z}_2(x,y)$ is said to be *quasi-normal* with respect to (50) if

(i) $AD - BC \neq 0$ on $\bar{G}$, $(A = \bar{\phi}_{p_1}, B = \bar{\phi}_{p_2}, C = \bar{\phi}_{q_1}, D = \bar{\phi}_{q_2}, E = \bar{\phi}_{z_1}, F = \bar{\phi}_{z_2})$;

(ii) The set where

$$\frac{\partial}{\partial y}\left[\frac{D(E-A_x-C_y)-C(F-B_x-D_y)}{AD-BC}\right]$$
$$=\frac{\partial}{\partial x}\left[\frac{A(F-B_x-D_y)-B(E-A_x-C_y)}{AD-BC}\right]$$

is nowhere dense on $\bar{G}$; and

(iii) If ζ_1 and ζ_2 are any functions of class C'' on $\bar{G}$ which satisfy (51) on $\bar{G}$ and vanish on G^*, there exists a one-parameter family $Z_1(x, y; \mu)$, $Z_2(x, y; \mu)$ of pairs of functions of class C'' in their arguments for (x, y) a point of $\bar{G}$ and $|\mu| < \mu_0$ such that

(a) $Z_1(x, y; 0) = \bar{z}_1(x, y)$, $Z_2(x, y; 0) = \bar{z}_2(x, y)$;

(b) $[\partial Z_1/\partial\mu]_{\mu=0} = \zeta_1$, $[\partial Z_2/\partial\mu]_{\mu=0} = \zeta_2$;

(c) for each μ with $|\mu| < \mu_0$, $Z_1(x, y; \mu)$ and $Z_2(x, y; \mu)$ coincide on G^* with $z_1(x, y)$ and $z_2(x, y)$, respectively; and

(d) $Z_1(x, y; \mu)$ and $Z_2(x, y; \mu)$ satisfy (50) for each μ with $|\mu| < \mu_0$.

THEOREM 1. *Let f, ϕ, G, $\bar{z}_1$, and $\bar{z}_2$ satisfy the differentiability hypotheses of the second paragraph of this section and suppose that the surface $z_1 = \bar{z}_1(x, y)$, $z_2 = \bar{z}_2(x, y)$ is a quasi-normal surface with respect to the differential equation* (50). *Suppose that $\bar{z}_1(x, y)$ and $\bar{z}_2(x, y)$ minimize the integral* (49) *among all pairs $z_1(x, y)$, $z_2(x, y)$ of class C'' on $\bar{G}$, coinciding with $\bar{z}_1$ and $\bar{z}_2$ on G^*, and satisfying* (50).

Then there exists a unique function, $\lambda(x, y)$ of class C' on $\bar{G}$ which satisfies the equations

$$(\partial/\partial x)(\bar{f}_{p_1} - \lambda\bar{\phi}_{p_1}) + (\partial/\partial y)(\bar{f}_{q_1} - \lambda\bar{\phi}_{q_1}) = \bar{f}_{z_1} - \lambda\bar{\phi}_{z_1},$$
$$(\partial/\partial x)(\bar{f}_{p_2} - \lambda\bar{\phi}_{p_2}) + (\partial/\partial y)(\bar{f}_{q_2} - \lambda\bar{\phi}_{q_2}) = \bar{f}_{z_2} - \lambda\bar{\phi}_{z_2}$$

on $\bar{G}$. (The notation of equation (51) is employed here.)

Proof. Let $\zeta_1(x, y)$ and $\zeta_2(x, y)$ satisfy the conditions (iii) of the definition. Since the surface $z_1 = \bar{z}_1(x, y)$, $z_2 = \bar{z}_2(x, y)$ is quasi-normal with respect to (50), there exists a one-parameter family of pairs of functions $Z_1(x, y; \mu)$, $Z_2(x, y; \mu)$ which satisfy the conditions (iii) of the definition. Let $I(\mu)$ be the function of μ obtained by substitution of the $Z_i(x, y; \mu)$ in the integral (49). Since $\bar{z}_1$ and $\bar{z}_2$ minimize this integral, we must have

$$I'(0) = \int_G\int (\bar{f}_{p_1}\zeta_{1x} + \bar{f}_{p_2}\zeta_{2x} + \bar{f}_{q_1}\zeta_{1y} + \bar{f}_{q_2}\zeta_{2y} + \bar{f}_{z_1}\zeta_1 + \bar{f}_{z_2}\zeta_2)\,dydx = 0.$$

That is to say, we have

$$\int_G\int (a\zeta_{1x} + b\zeta_{2x} + c\zeta_{1y} + d\zeta_{2y} + e\zeta_1 + f\zeta_2)\,dydx = 0$$

where $a = f_{p_1}$, etc., for every pair (ζ_1, ζ_2) of class C'' on $\bar{G}$, vanishing on G^*, and satisfying

$$A\zeta_{1x} + B\zeta_{2x} + C\zeta_{1y} + D\zeta_{2y} + E\zeta_1 + F\zeta_2 = 0$$

where $A = \bar{\phi}_{p_1}$, etc. The existence and uniqueness of $\lambda(x, y)$ follows from Lemma 16 of **3**, since it is clear that a, A, etc., are all of class C'''.

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THE NON-LOCAL EXISTENCE PROBLEM OF ORDINARY DIFFERENTIAL EQUATIONS.*

By Aurel Wintner.

Introduction. Let $f_1, \cdots, f_n$ be real-valued, continuous functions of position on the $(n+1)$-dimensional region

$$(1) \qquad -a \leqq t \leqq a, \qquad -\infty < x_i < \infty; \qquad (i = 1, \cdots, n).$$

Then, corresponding to any set of real values c_i, the system of differential equations

$$(2) \qquad x'_i = f_i(t; x_1, \cdots, x_n); \qquad (i = 1, \cdots, n),$$

where $x' = dx/dt$, has *near* $t = 0$ a solution $x_i = (t)$ satisfying

$$(3) \qquad x_i(0) = c_i.$$

The italicized proviso is essential, even if the system is "conservative" (in the sense that (1) and (2) become

$$(4) \qquad -\infty < x_i < \infty; \qquad (i = 1, \cdots, n)$$

and

$$(5) \qquad x'_i = f_i(x_1, \cdots, x_n)$$

respectively) and even if the functions f_i are regular-analytic on the whole x-space.

For instance, if (5) is the single differential equation

$$(6) \qquad x' = x^2,$$

then, although $f_1 = f_n$ is a polynomial, every solution $x(t)$ determined by a non-vanishing $x(0)$ ceases to exist at a finite $t = t_0$. In fact, all solutions distinct from the solution $x(t) \equiv 0$ (which belongs to $x(0) = 0$) are represented by $x(t) = (t_0 - t)^{-1}$, where t_0 is a non-vanishing integration constant determined by the initial value $x(0)$. But the function $(t_0 - t)^{-1}$ cannot be continued beyond $t = t_0$ so as to remain a solution. For, if it could, then,

* Received March 28, 1945.

since it should satisfy the differential equation, it ought to be defined at $t=t_0$ in such a way as to be differentiable, hence continuous, at $t=t_0$. But this is impossible, since the solution is not bounded near $t=t_0$.

These trivial comments are made only in view of (iii bis) below and in order to make it clear that the possibility of "moving *singularities*" in the particular case of *analytic* differential equations is just a manifestation of the *local* nature of the existence theorem in the *general* case. In this regard, not even Riccati's type, though reducible to a linear differential equation, is exceptional, since (6) is of his type.

On the other hand, the existence theorem of the initial problem (3) is not of a local nature if the system (2) is linear and, without additional loss of generality, homogeneous, that is, if

$$(7) \qquad f_i(t;x_1,\cdots,x_n)=f_{i1}(t)x_1+\cdots+f_{in}(t)x_n,$$

where, by virtue of the assumption made with regard to the range (1), the n^2 functions $f_{ik}(t)$ are continuous on the interval $-a\leqq t\leqq a$. In fact, the solution of (2) determined by arbitrary initial data (3) is known to exist on the whole interval $-a\leqq t\leqq a$ in the case (7). As was shown in [1], pp. 553-556, the non-local existence theorem for the linear case can be obtained as a *corollary* of the local existence theorem for the non-linear case, if the limited t-range supplied by the latter is applied repeatedly. The possibility of this reduction depended on the divergence of the harmonic series, $\Sigma 1/m=\infty$, a series resulting from the particular structure of (7).

Loc. cit., this elimination of the non-local existence theorem of linear systems as an *independent* theorem was just a matter of convenience, namely, a way of replacing two proofs by one. It took me twenty years to observe that the reduction process applied in [1] has more to it than this, just *methodical*, interest. In fact, it turns out that the whole of this reduction process can be worded so as to avoid the assumption of linearity. And it happens that the resulting rewording leads to *a solution of the non-local existence problem ("in the large," i. e., in the "unrestricted" sense of* [3], pp. 85-86, 132, 341) *for the non-linear equations* (2); *a solution which proves to be of a final nature.* Cf. (iii*) and (iii bis) below.

Both the result and its proof are very similar to the procedure applied by me to Strömgren's empirical principle of natural termination. On the other hand, an application of Birkhoff's subsequent variant of this proof of the termination principle would not lead to a solution of the problem (that is, to (iii*), (iii), or, for that matter, to the weaker results (i*), (i), which are not final in nature), since Birkhoff's variant consists in the replacement

of the *constructive* process of prolongation by the *indirect* device of the Heine-Borel theorem. For references cf. [3], pp. 441-442.

It should finally be mentioned that the proofs, which will be worded for the case of a real variable t, need no modification if t is replaced by $z = x + iy$; so that *all results remain valid in the case of (ordinary, non-linear) analytic differential equations in the complex domain.*

1. The arrangement of the proofs becomes particularly convenient if the most general case, instead of being formulated directly, is reached through successive stages of straightforward generalizations. In fact, all the essential elements of the proof are needed for the following particular case:

(i) *If* $f_1, \cdots, f_n$ *are continuous functions of position on the space* (4), *and if* $(c_1, \cdots, c_n)$ *is any point of this space, then* (5) *and* (3) *have a solution* $x_i = x_i(t)$ *which exists on the whole t-axis,* $-\infty < t < \infty$, *whenever the given functions* f_i *are subject to the n inequalities*

$$f_i(x_1, \cdots, x_n) = O(|x|) \text{ as } |x| \to \infty. \tag{8}$$

Here, and in the sequel, $|x|$ denotes, not the Euclidean length of the vector x, but the length of the longest component of x, that is,

$$|x| = \max(|x_1|, \cdots, |x_n|), \text{ where } x = (x_1, \cdots, x_n) \tag{9}$$

(this leads to $|x| = |x|$ if $n = 1$). Accordingly, (8) means the existence of a constant A satisfying

$$|f_i(x_1, \cdots, x_n)| < A \max(|x_1|, \cdots, |x_n|, 1). \tag{10}$$

The $(n+1)$-th term, 1, on the right of (10) is adjoined in order to take care of the vicinity of the origin of the x-space.

Since only the continuity of the functions f_i is assumed in (i), the solutions to which the assertion of (i) refers will not in general be unique. In addition, the solution can be unique near $t = 0$ but then lead to branch-points. What will happen in such a case is clear from the wording of (i) and from the following proof. Needless to say, the emphasis in (i) is not on this generality, since what really is interesting in (i) is the case of "smooth" functions f_i of the type occurring in dynamical applications (or, for that matter, the case of regular-analytic functions); cf. [3], p. 356. And the solutions must be unique *throughout* (and so, by (i), for $-\infty < t < \infty$), if the given functions f_i satisfy, for instance, the local condition of Lipschitz (e.g., if every f_i is of class C').

If the n continuous functions $f_i(x_1, \cdots, x_n)$ have an absolute value not exceeding M in the cube $|x_i - x_{i0}| \leqq b$ about a point $(x_{10}, \cdots, x_{n0})$, and if t_0 is arbitrary, then (5) has a solution $x_i(t)$ which satisfies the initial condition $x_i(t_0) = x_{i0}$ and exists, at least, on the t-interval consisting of those values which differ from t_0 by not more than b/M, and the n inequalities $|x_i(t) - x_{i0}| \leqq b$ hold for every t contained in this interval. The lower bound b/M for the t-range of assured existence follows by the method of equicontinuous functions. The same b/M results when, under an additional restriction of Lipschitz's type, another method, such as the method of successive approximations, is applied. Actually, *b/M cannot be improved in the complex-analytic case*, a case in which b/M becomes a "best universal constant" (of the type of absolute constants occurring in the theory of conformal mapping); cf. [2].

If the constants x_{i0}, t_0, b are identified with $c_i, 0, 1$ respectively, it is clear from the definition of M and from (10), that M can be chosen to be $A \max(c, 1) = Ac$, if c is any value exceeding each of the $n+1$ numbers $|c_i|, 1$. Hence, if c is fixed in this manner, then $b/M = 1/M$ becomes the reciprocal value of Ac. Consequently, if

$$t_1 = A^{-1}/c, \tag{11}$$

then (2) and (3) have a solution $x_i(t)$ on the interval $0 \leqq t \leqq t_1$, and the n inequalities $|x_i(t) - c_i| \leqq 1$ hold for every t contained in this interval and so, in particular, for $t = t_1$. It follows, therefore, from the definition of c that

$$|x_i(t_1)| < c + 1. \tag{12}$$

Next, let x_{i0}, t_0, b be identified with $x_i(t_1), t_1, 1$ respectively. Then it is clear from the definition of M (for the resulting cube) and from (10) and (12), that $A \max(c+1, 1) = A(c+1)$ is an admissible value of M. Hence, $b/M = 1/M$ becomes t_2, if t_2 denotes the quotient

$$t_2 = A^{-1}/(c+1). \tag{13}$$

Consequently, the solution $x_i(t)$, the existence of which was assured for $0 \leqq t \leqq t_1$, proves to be prolongable for $t_1 \leqq t \leqq t_1 + t_2$. Since $x_{i0} = x_i(t_1)$, it also follows that the n inequalities $|x_i(t) - x_i(t_1)| \leqq 1$ are satisfied for every t contained in the latter interval and so, in particular, for $t = t_1 + t_2$. Accordingly, from (12),

$$|x_i(t_1 + t_2)| < c + 2. \tag{14}$$

If x_{i0}, t_0, b are identified with $x_i(t_1 + t_2)$, $t_1 + t_2$, 1 respectively, it becomes clear that the possibility of repeating this procedure is never arrested. The m-th term of the sequence which starts with (11) and (13) is seen to be

$$t_m = A^{-1}/(c + m - 1), \tag{15}$$

and the m-th step extends the solution initiated by (3) to the t-interval

$$t_1 + \cdots + t_{m-1} \leqq t \leqq t_1 + \cdots + t_m, \qquad (t_0 = 0), \tag{16}$$

since, corresponding to (12) and (14),

$$| x_i(t_1 + \cdots + t_{m-1}) | < c + m - 1. \tag{17}$$

The first m intervals (16) cover the range $0 \leqq t \leqq T_m$, where

$$T_m = t_1 + \cdots + t_m. \tag{18}$$

But, since $\Sigma 1/m = \infty$, it is clear from (15) and (18) that $T_m \to \infty$ as $m \to \infty$. Hence, the existence domain of the solution $x_i(t)$ contains the whole half-axis $0 \leqq t < \infty$. Since t can be replaced by $-t$, the proof of (i) is now complete.

2. If the functions f_i are polynomials, then the assumption (8) of (i) is satisfied only if the system (5) is linear. On the other hand, (8) includes an extensive class of algebraic, and for that matter rational, functions f_i, as well as entire functions f_i (as exemplified by such types as

$$f(x) = (1 + x^3)^{-\frac{1}{3}}, \quad (1 + x^2)^{1/3}, \quad x^3/(1 + x^2), \quad x \exp \sin x,$$

where $n = 1$). In the linear case, the general solution of (5) consists of exponentials e^{ut} and (possibly) of $t, t^2, \cdots, t^{n-1}$, since t does not occur in (5). In particular, every solution is $O(e^{Ct})$, as $t \to \infty$, in this trivial case. In the non-linear case, no "explicit" solution of (5) is possible, of course (except when $n = 1$). But it turns out that *the growth of the solutions in the non-linear case* is limited by the same estimate as in the trivial case of a linear system with constant coefficients:

(ii) *Under the assumptions of* (i), *each of the n components $x_i(t)$ of every solution of* (5) *is $O(e^{Ct})$ as $t \to \infty$, where C is a constant. In addition, C can be chosen so as to be independent of the integration constants* (3).

In fact, it is clear from the proof of (17) that $| x_i(t) | < c + m$ holds for every positive t not exceeding the value (18). On the other hand, it is

seen from (15) that, as $m \to \infty$, the value (18) is asymptotically equal to $A^{-1} \log m$. This proves the first part of (ii). The refinement claimed by the second part follows by observing that the constant A, being defined by the formulation (10) of (8), depends only on the system (5).

The trivial case mentioned before (ii) shows that the estimate supplied by (ii) cannot be improved if only (8) is assumed. However, it is clear from the proof of (ii) that C in (ii) can be chosen arbitrarily small if (10) holds for every A. In other words, (ii) can be supplemented as follows:

(ii bis) *If* $O(|x|)$ *in the assumption* (8) *of* (i) *is refined to* $o(|x|)$, *then* $O(e^{Ct})$ *in the assertion of* (ii) *can be improved to* $\exp o(t)$.

By an adaptation of these remarks, estimates of the growth which correspond to (ii), (ii bis) can readily be obtained for each of the cases covered by the general result (iii) below.

3. In (i), the differential equations are of the particular type (5), that is, t does not occur explicitly. In order to extend (i) to the general case (2), let $f_1, \cdots, f_n$ be given as continuous functions of position on the $(n+1)$-dimensional region (1). The t-bound b/M supplied by the local existence theorem, used in the above proof of (i), must then be replaced by $\min(a, b/M)$, where a is the number occurring in (1). Correspondingly, the values (11), (13), (15) must now be restricted so as to be within the t-range of (1), that is, (15) must be replaced by

$$t_m = \min(a, A^{-1}/(c+m-1)). \tag{19}$$

Then the extension of the solution to the consecutive intervals (16) is exactly the same as in the proof of (i), if it is assumed that the functions f_i (which now contain t) satisfy the inequalities (10), where A is a constant (independent of $(x_1, \cdots, x_n)$ *and* of t).

Thus, under the assumption just mentioned, the existence of the solution $x_i(t)$ follows on the interval $0 \leqq t \leqq T_m$, where m is arbitrary and T_m is defined by (18) and (19). However, since $\Sigma 1/m = \infty$, it is clear from (18) and (19) that T_m becomes identical with a when m is large enough. Consequently, (i) can be extended from (5) to (2), as follows:

(i*) *If* $f_1, \cdots, f_n$ *are continuous functions of the position* $(t; x_1, \cdots; x_n)$ *on the domain* (1), *and if* $c_1, \cdots, c_n$ *are arbitrary, then* (2) *and* (3) *have a solution* $x_i = x_i(t)$ *on the whole t-interval,* $-a \leqq t \leqq a$, *whenever the given functions are subject to the n inequalities*

$$(20) \qquad \max_{-a \leqq t \leqq a} | f_i(t; x_1, \cdots, x_n) | = O(|x|)$$

as (9) *tends to* ∞.

Clearly, (i*) implies the classical result as to the unrestricted existence of the solutions of linear differential equations. In fact, (20) is satisfied in the case (7), where the $f_{ik}(t)$ are arbitrary continuous functions on the interval $-a < t \leqq a$. Conversely, since the estimate (20) is just what is required in the case of linear systems, the exceptional function-theoretical standing of the linear systems might suggest that (20) cannot be relaxed. However, it turns out that (20) is not the true condition, since (i*) can be improved as follows:

(iii*) *The assertion of* (i*) *remains valid if* $O(|x|)$ *in* (20) *is relaxed to* $O(L(|x|))$, *where* $L(r)$ *is any L-function satisfying*

$$(21) \qquad \int^{\infty} dr/L(r) = \infty.$$

This contains (i*), since $\int dr/L(r) = \log r$ in the case, $L(r) = r$, of (20). And (iii*) is the true theorem, since its integral condition cannot be altered, as will be shown below.

4. The proof of (i), where (2) is of the form (5), is based on the fact that the series $\Sigma 1/m$ diverges. Correspondingly, the proof of (i) remains unaltered in cases in which $\Sigma 1/(m \log m)$ or $\Sigma 1/(m \log m \log\log m)$ takes the place of $\Sigma 1/m$. On the other hand, what introduced the particular series $\Sigma 1/m$ in the proof of (i) was just the linearity of the function, (9), occurring in the O-assumption, (8), of (i). Thus it is clear that (i) can be generalized as follows:

(iii) *The assertion of* (i) *remains valid if* $O(|x|)$ *on the right of* (8) *is relaxed to* $O(L(|x|))$, *where* $L(r)$ *is any L-function satisfying* (21).

Actually, the assumption that $L(r)$ be an L-function is not needed to its full extent. On the other hand, the assumption (21) is fundamental indeed:

(iii bis) *Corresponding to any L-function subject to*

$$(21\text{ bis}) \qquad \int^{\infty} dr/L(r) < \infty,$$

there exist systems (5), *even analytic systems, which satisfy the assumptions of* (iii) *except the assumption* (21), *but are such as to make the assertion of* (iii) *false* (*i. e., such as to possess solutions* $x_i(t)$ *which cease to exist at a finite* $t = t^0$).

In order to see this, let $n = 1$, and let $f_1(x_1) = f(x)$ be identical with $L(x)$ (for large positive $x_1 = x$), where $L(r)$ is any given L-function satisfying (21 bis). Thus the system (5) becomes the single equation $x' = L(x)$ (for large $x > 0$). Its solution $x = x(t)$ results by the inversion of the relation

$$t = \int_c^x dr/L(r) + \text{const.} \tag{22}$$

(if c is large enough). Since $L > 0$, the integral (22) is an increasing function of x $(> c)$, and so the correspondence (22) between t and x is one-to-one. But it is clear from (22) and (21 bis) that t tends to a finite limit as $x \to \infty$. Hence, if t^0 denotes this limit, then $x(t) \to \infty$ as $t \to t^0$, where $t < t_0$ $(< \infty)$. It follows therefore from the remarks made after (6), that the solution $x(t)$ cannot be prolonged beyond t^0.

This proves (iii bis). And (iii bis) implies the assertion made after (iii*). Finally, in order to prove (iii*), it suffices to make in the proof of (iii) the trivial modification which led from the proof of (i) to that of (i*).

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CRITERIA FOR THE ABSOLUTE CONVERGENCE OF A FOURIER SERIES AT A GIVEN POINT.*

By KIEN-KWONG CHEN.

1. In the case of the convergence of the Fourier series of f at a given point, the problem is entirely controlled by the behavior of f in the neighborhood of that point. But the absolute convergence of the series is not a local property, and in fact depends on the behavior of the function throughout the whole interval of definition.

Various sufficient conditions for the absolute convergence of the series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

are known.[1] Any one of these implies the convergence of the series

$$\sum (|a_n| + |b_n|). \tag{2}$$

The necessary and sufficient condition for the absolute convergence of a trigonometrical series in the whole interval, is that the series is a Fourier series of a Young's continuous function.[2]

2. Let $h(x)$ be an even function such that $h(x) = x^a$ $(0 < a < 1)$ for $0 \leqq x \leqq \pi$, and that $h(x + 2\pi) = h(x)$ for every x. Write

$$g(x) = \sum_{n=1}^{\infty} (\sin nx)/n, \qquad h(x) + g(x) = f(x),$$

and suppose that the Fourier series of $f(x)$ is (1). Then we have $b_n = 1/n$. An easy calculation shows that $a_n = O(n^{-1-a})$. Thus, the Fourier series of f converges absolutely at the point

$$x = 0, \tag{3}$$

without the convergence of the series (2). It is perhaps worth remarking that the example may be obtained more quickly by setting $a_n = 0$, $b_n = 1/n$.

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[1] Cf. A. Zygmund (1), Chapter VI.

[2] K. K. Chen (1); G. H. Hardy and J. E. Littlewood (1).

The absolute convergence at the origin (3) of the Fourier series can however be known by the following criterion.[3]

THEOREM 1. *If $0 < p < 1$, and the derivative of*

$$\int_0^t \frac{u^p \phi(u)\,du}{(t-u)^p} \tag{4}$$

exists as a function of bounded variation i nthe interval $0 \leqq t \leqq \pi$, where

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\} \tag{5}$$

then the Fourier series of f converges absolutely at the point x.

In fact, when $\phi(t) = \frac{1}{2}t^a$, the derivative of the function (4) becomes a constant multiple of t^a which is of bounded variation in $(0, \pi)$, since a is non-negative.

3. In order to prove Theorem 1, we establish several lemmas.

Suppose that $0 < \alpha < 1$, $x \geqq 0$, and that the integral of h from 0 to x, of order $1 - \alpha$

$$(h(x))_{1-\alpha} = (1/\Gamma(1-\alpha)) \int_0^x (x-t)^{-\alpha} h(t)\,dt$$

has the derivative $(d/dx)(h(x))_{1-\alpha}$ at the point x, then we write

$$(h(x))_{-\alpha} = (d/dx)(h(x))_{1-\alpha}.$$

LEMMA 1. *If the derivative of $(h(x))_{1-\alpha}$ $(0 < \alpha < 1)$ exists as a bounded function in $(0, X)$, then $(h(x))_\alpha$ is equivalent to a function of* Lip α *and*

$$h(x) = ((h(x))_{-\alpha})_\alpha, \qquad (0 \leqq x \leqq X),$$

except for a null set of points x.

Proof. Write

$$(d/dx)(h(x))_{1-\alpha} = g(x), \qquad \max_{0 \leqq x \leqq X} |g(x)| = A.$$

If $0 \leqq x < x' \leqq X$, then we have

$$|(h(x))_{1-\alpha} - (h(x'))_{1-\alpha}| \leqq A(x' - x).$$

Hence $(h(x))_{1-\alpha}$ is an absolutely continuous function.

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[3] Another criterion for the absolute convergence of a Fourier series at a fixed point can be found in the note: K. K. Chen (2).

Since $g(t)$ is bounded, $(g(x))_\alpha$ is continuous.[4] Remembering that $0 < \alpha < 1$, we have

$$(h(x))_{1-\alpha} = \int_0^x g(t)\,dt = ((g(x))_\alpha)_{1-\alpha}.$$

It follows that the two functions $h(x)$ and $(g(x))_\alpha$ are equivalent. Hence $h(x)$ belongs to the Lebesgue class $L^p(0, X)$ for every $p > 1$. Hardy and Littlewood have proved that[5] if $p \geqq 1$, $0 < \alpha < 1$, and $h(x)$ belongs to L^p, then

$$\left[\int_0^X |(h(x))_\alpha - (h(x-\delta))_\alpha|^p\,dx\right]^{1/p} = o(\delta^\alpha),$$

as $\delta \to +0$. It can be deduced from this theorem that

$$(h(x))_\alpha \in \operatorname{Lip} \alpha. \tag{6}$$

But it is better to give a direct proof for (6). We have

$$\Gamma(\alpha)\{(h(x'))_\alpha - (h(x))_\alpha\} \tag{7}$$
$$= \int_0^x \{(x'-t)^{\alpha-1} - (x-t)^{\alpha-1}\} h(t)\,dt + \int_0^{x'} (x'-t)^{\alpha-1} h(t)\,dt.$$

We may suppose that $h(t)$ is continuous, so that

$$|h(t)| \leqq M\Gamma(\alpha+1) \qquad (0 \leqq t \leqq X),$$

if M is sufficiently large. Assuming $x' > x$, it follows from (7) that

$$|(h(x'))_\alpha - (h(x))_\alpha| \leqq \alpha M \int_0^x \{(x-t)^{\alpha-1} - (x'-t)^{\alpha-1}\}\,dt + M(x'-x)^\alpha$$
$$= M(x'^\alpha - x^\alpha) + 2M(x'-x)^\alpha \leqq 3M(x'-x)^\alpha,$$

since $0 < \alpha < 1$. This proves (6).

Finally, by definition, $(h(x))_{-\alpha} = g(x)$. Hence

$$((h(x))_{-\alpha})_\alpha = (g(x))_\alpha.$$

Since $((g(x))_\alpha$ is equivalent to $h(x)$, the proof is completed.

4. LEMMA 2. *Suppose that $q + 1 > \alpha > 0$, $\alpha < 1$, $0 < w \leqq \pi$, and that*

$$Z(w) = \int_0^w u^{q-\alpha} \int_u^\pi (t-u)^{\alpha-1} t^{-q} \cos nt\,dt\,du.$$

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[4] G. H. Hardy (1).

[5] G. H. Hardy and J. E. Littlewood (2).

Then there is a constant C such that

$$(8) \qquad | nZ(w) | \begin{cases} \leqq C(nw)^{-\alpha} & (nw \geqq 1), \\ \leqq C(nw)^{1-\alpha} & (nw < 1,\ \alpha \leqq q), \\ \leqq C(nw)^{1-\alpha+q} & (nw < 1,\ \alpha > q). \end{cases}$$

Proof. In the first place, it follows from

$$\begin{aligned} Z(\pi) &= \int_0^\pi t^{-q} \cos nt \int_0^t u^{q-\alpha}(t-u)^{\alpha-1} du\, dt \\ &= B(\alpha, q-\alpha+1) \int_0^\pi \cos nt\, dt \\ &= 0 \end{aligned}$$

that

$$\begin{aligned} Z(w) &= Z(\pi) - \int_w^\pi u^{q-\alpha} \int_u^\pi (t-u)^{\alpha-1} t^{-q} \cos nt\, dt\, du \\ &= -\int_w^\pi t^{-q} \cos nt \int_w^\pi u^{q-\alpha}(t-u)^{\alpha-1} du\, dt \\ (9) \qquad &= -\int_w^\pi \cos nt \int_{w/t}^1 v^{q-\alpha}(1-v)^{\alpha-1} dv\, dt \\ &= (1/n) \int_w^\pi \sin nt \cdot (w/t^2)(w/t)^{q-\alpha}(1-w/t)^{\alpha-1} dt \\ &= (1/n) w^{1+q-\alpha} \int_w^\pi (t-w)^{\alpha-1} t^{-1-q} \sin nt\, dt. \end{aligned}$$

The second mean value theorem gives two numbers t_1 and t_2 such that

$$nw^\alpha Z(w) = \int_w^{t_1} (t-w)^{\alpha-1} \sin nt\, dt, \qquad w < t_1 < \pi,$$

and that $w < t_2 < t_1$,

$$| nw^\alpha Z(w) | \leqq \int_w^{w+1/n} (t-w)^{\alpha-1} dt + | n^{1-\alpha} \int_w^{t_2} \sin nt\, dt |.$$

Hence

$$| nw^\alpha Z(w) | \leqq (2 + 1/\alpha) n^{-\alpha}.$$

This proves (8) for the case $nw \geqq 1$.

Secondly, let $nw \leqq 1$; we have

$$\begin{aligned} & w^{1+q-\alpha} \int_w^\pi (t-w)^{\alpha-1} t^{-1-q} \sin nt\, dt \\ &= (nw)^{1+q-\alpha} \int_0^n y^{\alpha-1} \frac{\sin(y+nw)}{(y+nw)^{1+q}} dy \\ &= (nw)^{1+q-\alpha} \Big(\int_0^{\frac{1}{2}\pi-1} + \int_{\frac{1}{2}\pi-1}^{n(\pi-w)} \Big) \\ &= (nw)^{1+q-\alpha} \{ \lambda_n \int_0^{\frac{1}{2}\pi-1} y^{\alpha-1}(y+nw)^{-q} dy + \mu_n \int_{\frac{1}{2}\pi-1}^\infty y^{\alpha-q-2} dy \}, \end{aligned}$$

where $|\lambda_n| < 1$, $|\mu_n| < 1$. Since $q + 1 > \alpha$, the integral over $(\pi/2 - 1, \infty)$ is convergent. The integral

$$I = \int_0^{\frac{1}{2}\pi - 1} y^{\alpha-1}(y + nw)^{-q} dy$$

is $O(1)$, when $\alpha > q$. And if $\alpha \leqq q$, then

$$I < (nw)^{-q} \cdot (1/\alpha)(\pi/2)^{\alpha}.$$

This completes the proof of the lemma.

5. LEMMA 3. *If $P(t)$ is of bounded variation in $(0, \pi)$ then the series*

$$\sum_{n=1}^{\infty} n^{-1} \int_0^{1/n} (nt)^{\epsilon} \, | d\phi(t) | \quad \textit{and} \quad \sum_{n=1}^{\infty} n^{-1} \int_{1/n}^{\pi} (nt)^{-\epsilon} \, | d\phi(t) |$$

are both convergent, where $\epsilon > 0$.

Proof. The function

$$H(t) = \sum_{n=1}^{\infty} n^{-1} \min \{ (nt)^{\epsilon}, (nt)^{-\epsilon} \}$$

is bounded in $(0, \pi)$. In fact,

$$H(t) = \sum_{nt \leqq 1} n^{-1}(nt)^{\epsilon} + \sum_{nt > 1} n^{-1}(nt)^{-\epsilon}.$$

This is clearly $O(1)$. The sum of any one of the two series in the lemma, is not greater than $\int_0^{\pi} H(t) \, | d\phi(t) |$ which is a finite number, since $\phi(t)$ is of bounded variation. Lemma 3 is thus proved.

6. Guided by these lemmas, we were led to the following theorem which is an extension of Theorem 1.

THEOREM 2. *If $0 < \alpha < 1$, $q \geqq \alpha$, and the function*

$$\psi(t) = t^{\alpha - q} \frac{d}{dt} \int_0^t \frac{u^q \phi(u) \, du}{(t - u)^{\alpha}} \tag{10}$$

exists and is of bounded variation in $(0, \pi)$, then the Fourier series of f converges absolutely at the point x.

This reduces to Theorem 1, when $\alpha = q = p$.

Proof. We have to deduce the absolute convergence of the series

$$\sum_{n=1}^{\infty} (2/\pi) \int_0^{\pi} \phi(t) \cos nt dt \tag{11}$$

from the fact that the function $\psi(t)$ is of bounded variation in $(0,\pi)$.[6]

In virtue of $q \geqq \alpha$, the function

$$(t^q\phi(t))_{-\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{u^q\phi(u)\,du}{(t-u)^\alpha}$$

being the product of $\psi(t)$ and $t^{q-\alpha}/\Gamma(1-\alpha)$, is of bounded variation in $(0,\pi)$.

It follows from Lemma 1 that the relation

$$\phi(t) = t^{-q}((t^q\phi(t))_{-\alpha})_\alpha$$

holds almost everywhere. Therefore the n-th term of (11) can be written as

$$\int_0^\pi t^{-q} \cos nt \cdot (1/\Gamma(\alpha)) \int_0^t (t-u)^{\alpha-1}(u^q\phi(u))_{-\alpha}\,du\,dt$$

$$= (1/\Gamma(\alpha)) \int_0^\pi (u^q\phi(u))_{-\alpha} \int_u^\pi (t-u)^{\alpha-1}t^{-q} \cos nt\,dt\,du$$

$$= (1/\Gamma(\alpha)) \int_0^\pi u^{\alpha-q}(u^q\phi(u))_{-\alpha}\,dZ(u).$$

On integration by parts, we obtain

$$\Gamma(\alpha)\Gamma(1-\alpha) \int_0^\pi \phi(t) \cos nt\,dt = -\int_0^\pi Z(w)\,d\psi(w)$$

since $Z(0) = Z(\pi) = 0$. Hence, by Lemma 2,

$$\left| \int_0^\pi \phi(t) \cos nt\,dt \right| \leqq (C/n) \int_0^{1/n} (nw)^{1-\alpha} \,|\, d\psi(w) \,| + (C/n) \int_{1/n}^\pi (nw)^{-\alpha} \,|\, d\psi(w) \,|. \tag{12}$$

The absolute convergence of (11) follows from (12) and Lemma 3. The theorem is thus proved.

7. If we equate α and q in Theorem 2, we obtain Theorem 4. In particular, if the derivative $\phi'(t)$ exists and both $\phi(t)$ and $t\phi'(t)$ are of bounded variation in $(0,\pi)$, then

$$\psi(t) \equiv \frac{d}{dt} \int_0^t \frac{u^p\phi(u)\,du}{(t-u)^p} = \frac{d}{dt} \int_0^1 \frac{tv^p\phi(tv)\,dv}{(1-v)^p}$$

$$= \int_0^1 \frac{v^p(\phi(tv) + tv\phi'(tv))\,dv}{(1-v)^p}.$$

[6] Cf. L. S. Bosanquet (1).

which is evidently also of bounded variation in $(0, \pi)$. Hence we obtain the following

THEOREM 3. *If $\phi(t)$ and $t\phi'(t)$ are both of bounded variation in $(0, \pi)$ then the Fourier series of f converges absolutely at the point x.*

8. Write $\sigma_n{}^\alpha$ for the n-th Cesàro mean of order α of the series $\sum_{n=0}^{\infty} a_n$. Thus, writing $(\alpha)_n = \dfrac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}$ for $\alpha > -1$,

$$\sigma_n{}^\alpha = (1/(\alpha)_n) \sum_{\nu=0}^{\infty} (\alpha)_{n-\nu} a_\nu.$$

The series Σa_n is said to be absolutely summable (C, α) or summable $| C, \alpha |$, if the series

$$\sum_{n=0}^{\infty} | \sigma_n{}^\alpha - \sigma_{n-1}{}^\alpha |$$

with $\sigma_{-1}{}^\alpha = 0$, converges. In this case, letting $\sigma_n{}^\alpha \to s$, we write

$$\sum_{n=0}^{\infty} a_n = s \mid C, \alpha \mid,$$

In particular, summability $| C, 0 |$ is the same as absolute convergence.

LEMMA 4. *If $\alpha > -1$ and $\sum a_n = s \mid C, \alpha \mid$, then for $\epsilon > 0$,*

$$\sum a_n = s \mid C, \alpha + \epsilon \mid.$$

This important theorem is due to Kogbetliantz.[7] It seems however that the proposition is new in the case $-1 < \alpha < 0$. We give a proof here.

Proof. Without loss of generality, we can assume that $\epsilon < 1$. Writing $\alpha + \epsilon = \beta$, we have, by Abel's transformation,

$$\begin{aligned}(\beta)_n \sigma_n{}^\beta &= \sum_{\nu=0}^{n} (\epsilon - 1)_{n-\nu} (\alpha)_\nu \sigma_\nu{}^\alpha \\ &= \sum_{\nu=0}^{n-1} (\sigma_\nu{}^\alpha - \sigma_{\nu+1}{}^\alpha) \sum_{\mu=0}^{\nu} (\alpha)_\mu (\epsilon - 1)_{n-\mu} + \sigma_n{}^\alpha (\beta)_n.\end{aligned}$$

It follows that

$$\sigma_n{}^\beta - \sigma_{n-1}{}^\beta = ((\alpha)_n/(\beta)_n)(\sigma_n{}^\alpha - \sigma_{n-1}{}^\alpha) + \sum_{\nu=0}^{n-2} (\sigma_\nu{}^\alpha - \sigma_{\nu+1}{}^\alpha)(n, \nu),$$

[7] E. Kogbetliantz (1), (2).

where

$$(n,\nu)=\sum_{\mu=0}^{\nu}(\alpha)_\mu\left\{\frac{(\epsilon-1)_{n-\mu}}{(\beta)_n}-\frac{(\epsilon-1)_{n-\mu-1}}{(\beta)_{n-1}}\right\}=(n,\nu)_1+(n,\nu)_2,$$

$$(n,\nu)_1=\frac{1+\alpha}{1-\epsilon}\,\frac{1}{(\beta)_{n-1}}\sum_{\mu=0}^{\nu}(\alpha)_\mu(\epsilon-2)_{n-\mu},$$

$$(n,\nu)_2=\frac{\beta(\alpha+1)}{(\epsilon-1)(\beta+1)(\beta+1)_{n-1}}\sum_{\mu=0}^{\nu}(\alpha+1)_\mu(\epsilon-2)_{n-\mu}.$$

If $N>2$, then we have

$$\sum_{n=1}^{N}|\sigma_n{}^\beta-\sigma_{n-1}{}^\beta|\leqq\sum_{n=1}^{N}((\alpha)_n/(\beta)_n)\,|\sigma_n{}^a-\sigma_{n-1}{}^a|+\sum_{n=2}^{N}\sum_{\nu=0}^{n-2}|((\sigma_\nu{}^a-\sigma_{\nu+1}{}^a)\,(n,\nu)|.$$

The first term is $O(1)$, as $N\to\infty$, since $(\beta)_n>(\alpha)_n$. The last term is equal to

$$\sum_{\nu=0}^{N-2}|\sigma_\nu{}^a-\sigma_{\nu+1}{}^a|\sum_{n=\nu+2}^{N}|(n,\nu)|.$$

Hence it is enough to show that $\sum_{\nu+2}^{N}|(n,\nu)|$ is uniformly bounded in N and ν.

There is a constant K such that

$$\sum_{n=2\nu+1}^{N}|(n,\nu)_1|<K\sum_{n=2\nu+1}^{N}n^{-\beta}\cdot n^{\epsilon-2}\sum_{\mu=0}^{\nu}(\alpha)_\mu.$$

This is equal to

$$K(\alpha+1)_\nu\sum_{n=2\nu+1}^{N}n^{-2-\alpha}=O(1),$$

uniformly in N and ν.

If $\nu+2\leqq n\leqq 2\nu$, then $(\beta)_{n-1}=O(\nu^\beta)$ and

$$\begin{aligned}\sum_{n=\nu+2}^{2\nu}\sum_{\mu=0}^{\nu}(\alpha)_\mu|(\epsilon-2)_{n-\mu}|&=-\sum_{n=\nu+2}^{2\nu}\sum_{\mu=0}^{\nu}(\alpha)_\mu(\epsilon-2)_{n-\mu}\\&=-\sum_{\mu=0}^{\nu}(\alpha)_\mu\sum_{n=\nu+2}^{2\nu}(\epsilon-2)_{n-\mu}\\&=\sum_{\mu=0}^{\nu}(\alpha)_\mu((\epsilon-1)_{\nu-\mu+1}-(\epsilon-1)_{2\nu-\mu}).\end{aligned}$$

This is less than $\sum_{\mu=0}^{\nu}(\alpha)_\mu(\epsilon-1)_{\nu-\mu}=(\beta)_\nu$. Hence

$$\sum_{n=\nu+2}^{2\nu}(1/(\beta)_{n-1})\sum_{\mu=0}^{\nu}(\alpha)_\mu(\epsilon-2)_{n-\mu}=O(\nu^{-\beta})\,(\beta)_\nu=O(1).$$

Therefore we have

$$\sum_{n=\nu+1}^{\infty} |(n, \nu)_1| = O(1).$$

In a similar manner, we can prove

$$\sum_{n=\nu+1}^{\infty} |(n, \nu)_2| = O(1).$$

The proposition is thus established.

9. It has now been shewn that the absolute summability of negative order is a property which is stronger than absolute convergence. Let us next consider the absolute summability of negative order. For this purpose, we write

$$[\chi(t)]_\alpha = \Gamma(1+\alpha) t^{-\alpha} (\chi(t))_\alpha$$

as the mean function of order α of $\chi(t)$. Thus

$$[\chi(t)]_0 = \chi(t),$$
$$[\chi(t)]_\alpha = (\alpha/t^\alpha) \int_0^t (t-u)^{\alpha-1} \chi(u)\, du \qquad (\alpha > 0,\ t > 0),$$

provided that $\chi(t)$ is integrable in the Lebesgue sense over $(0, t_0)$. The latter equation in the case $0 < \alpha < 1$ holds good for almost all values of t in $(0, t_0)$.[8] The mean functions of negative order exist with additional hypotheses.

THEOREM 4. *Suppose that* $-1 < -\alpha < 0$, $q \geqq \alpha$ *and that at the point* x *the function*

(13) $$t^{-q}[t^q \phi(t)]_{-\alpha}$$

is of bounded variation in $(0, \pi)$; *then, for* $\epsilon > 0$, *the Fourier series of* f *is summable* $|C, -\alpha + \epsilon|$.

Before giving the proof of this theorem it is convenient to prove the following three lemmas.

10. LEMMA 5. *Write* $g^\alpha(n, t)$ *for the n-th Cesàro mean of order* α *of* $2\pi^{-1} \sin nt$. *If* $-1 < \beta < 0$,[9] *then*

$$|g^\beta(n, t)| \leqq Ant(1+nt)^{-1-\beta}, \qquad (t > 0).$$

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[8] G. H. Hardy (1).

[9] If $\alpha \geqq 0$, then $g^\alpha(n, t) = O(1+nt)^{-\alpha} + O(1+nt)$. See Obreschkoff (1).

Proof. In fact, we have

$$g^\beta(n,t) = \frac{2}{\pi}\frac{1}{(\beta)_n}\sum_{\nu=0}^{n}(\beta-1)_{n-\nu}\sin\nu t.$$

Abel's transformation gives

$$\sum_{\nu=0}^{n}(\beta-1)_{n-\nu}\sin\nu t = \sum_{\nu=0}^{n-1}(\beta)_\nu\Delta\sin(n-\nu)t$$
$$= -2\sin(t/2)\sum_{\nu=0}^{n-1}(\beta)_\nu\cos(n-\nu-\tfrac{1}{2})t$$

which is numerically less than a constant multiple of $tn^{1-\beta}$. This proves the lemma for the case $nt \leqq 1$. To complete the proof, we rewrite

$$\sum_{\nu=0}^{n}(\beta-1)_{n-\nu}\sin\nu t = \Big(\sum_{\nu t\leqq 1} + \sum_{nt>1}\Big)(\beta-1)_\nu\sin(n-\nu)t.$$

Now, if $nt > 1$,

$$\Big|\sum_{\nu t>1}(\beta-1)_\nu\sin(n-\nu)t\Big| \leqq \sum_{\nu>1/t}^{\infty}|(\beta-1)_\nu| \leqq At^{-\beta},$$

$$\sum_{\nu t\leqq 1}(\beta-1)_\nu\sin(n-\nu)t = -2\sin(t/2)\sum_{\nu t\leqq 1}(\beta)_\nu\cos(n-\nu-\tfrac{1}{2})t + O(t^{-\beta})$$
$$= O(t^{-\beta}).$$

Therefore we obtain

$$g^\beta(n,t) = O(nt)^{-\beta}.$$

The lemma is thus proved.

From the proof of Lemma 5, we can state the following result.

LEMMA 6. *If* $-1 < \beta < 0$, $nt \geqq 1$, *then the relation*

$$(1/(\beta)_n)\sum_{\nu t\leqq 1}^{\mu}(\beta-1)_\nu\sin(n-\nu)t = O(nt)^{-\beta}$$

holds uniformly for $t^{-1} \leqq \mu \leqq n$.

11. LEMMA 7. *Suppose that* $-1 < -\alpha < \beta < 0$, $1 + q > \alpha - \beta$ *and that*

$$Z_\beta(w) = \int_0^w u^{q-\alpha}\int_u^\pi (t-u)^{\alpha-1}t^{-q}(d/dt)g^\beta(n,t)\,dt\,du. \tag{14}$$

Then for $0 < w \leqq \pi$,

$$|Z_\beta(w)| \leqq (nw)^{-\beta}(1+nw)^{-\alpha}. \tag{15}$$

Proof. As in the proof of Lemma 2, we have

$$Z_\beta(\pi) = B(\alpha, q - \alpha + 1) \int_0^\pi (d/dt) g^\beta(n, t)\, dt = 0.$$

Hence

$$(16) \qquad Z_\beta(w) = -\int_w^\pi (d/dt) g^\beta(n, t) \int_{w/t}^1 v^{q-\alpha}(1 - v)^{\alpha-1} dv dt$$
$$= w^{1+q-\alpha} \int_w^\pi (t - w)^{\alpha-1} t^{-1-q} g^\beta(n, t)\, dt.$$

Write $g^\beta(n, t) = g_1(n, t) + g_2(n, t)$ with

$$(\beta)_n g_1(n, t) = \sum_{\nu=0}^{[n/2]} (\beta - 1)_\nu \sin (n - \nu) t.$$

Firstly, assuming $nw \geqq 2$, let us consider the following two integrals

$$Z_\beta{}^j(w) = w^{1+q-\alpha} \int_w^\pi (t - w)^{\alpha-1} t^{-1-q} g_j(n, t)\, dt; \qquad (j = 1, 2).$$

By Lemma 6, we have

$$(17) \qquad w^{1+q-\alpha} \int_w^{w+1/n} (t - w)^{\alpha-1} t^{-1-q} g_1(n, t)\, dt$$
$$= w^{1+q-\beta} O(nw)^{-\beta} \int_w^{w+1/n} (t - w)^{\alpha-1} t^{-1-q} dt = O(nw)^{-\alpha-\beta}.$$

The second mean value theorem gives a number w_1 for which $w < w_1 < 2w$, such that

$$w^{1+q-\alpha} \int_{w+1/n}^{2w} (t - w)^{\alpha-1} t^{-1-q} g_1(n, t)\, dt = O(n)(nw)^{-\alpha} \int_{w+1/n}^{w_1} g_1(n, t)\, dt.$$

We have

$$\int_{w+1/n}^{w_1} g_1(n, t)\, dt = \frac{1}{(\beta)_n} \left[\sum_{\nu=0}^{n/2} (\beta - 1)_\nu \frac{-\cos(n - \nu) t}{n - \nu} \right]_{w+1/n}^{w_1}.$$

Since $t^{-1} < n/2$, we have

$$\sum_{\nu t \leqq 1} (\beta - 1)_\nu \frac{\cos(n - \nu) t}{n - \nu} = \sum_{\nu t > 1} (\beta)_\nu \Delta \frac{\cos(n - \nu) t}{n - \nu} + O(n^{-1} t^{-\beta}).$$

And in virtue of $\beta < 0$, we have

$$\sum_{\nu t > 1} (\beta - 1)_\nu \frac{\cos(n - \nu) t}{n - \nu} = O(n^{-1} t^{-\beta}).$$

Therefore we obtain

(18) $$w^{1+q-\alpha}\int_{w+1/n}^{2w}(t-w)^{\alpha-1}t^{-1-q}g_1(n,t)\,dt = O(nw)^{-\alpha-\beta}.$$

Integrating by parts, we have

$$\int_{2w}^{\pi}(t-w)^{\alpha-1}t^{-1-q}g_1(t)\,dt$$

$$=\Big[(t-w)^{\alpha-1}t^{-1-q}\sum_{\nu=0}^{n/2}\frac{(\beta-1)_\nu}{(\beta)_n}\,\frac{-\cos(n-\nu)t}{n-\nu}\Big]_{2w}^{\pi}$$

$$+\int_{2w}^{\pi}\frac{d}{dt}\{(t-w)^{\alpha-1}t^{-1-q}\}\sum_{\nu=0}^{n/2}\frac{(\beta-1)_\nu}{(\beta)_n}\,\frac{\cos(n-\nu)t}{n-\nu}\,dt$$

$$=O(n^{-1-\beta})+O(w^{\alpha-q-2}n^{-1-\beta})+\int_{2w}^{\pi}-\frac{d}{dt}\{(t-w)^{\alpha-1}t^{-1-q}\}O(n^{-1})(nt)^{-\beta}dt,$$

since, for $t > w$,

$$-(d/dt)\{(t-w)^{\alpha-1}t^{-1-q}\} = t^{-2-q}(t-w)^{\alpha-2}\{(2+q-\alpha)t-(1+q)w\}$$
$$> t^{-2-q}(t-w)^{\alpha-2}(1-\alpha)w > 0.$$

Now, we have

$$0<\int_{2w}^{\pi}-(d/dt)\{(t-w)^{\alpha-1}t^{-1-q}\}\cdot t^{-\beta}dt$$
$$=O(1)+w^{\alpha-q-\beta-2}-\beta\int_{2w}^{\pi}(t-w)^{\alpha-1}t^{-2-q-\beta}dt = O(w^{\alpha-q-\beta-2}),$$

since $q+2>\alpha-\beta$. It follows that

(19) $$w^{1+q-\alpha}\int_{2w}^{\pi}(t-w)^{\alpha-1}t^{-1-q}g_1(t)\,dt = O(nw)^{-1-\beta}.$$

Collecting (17), (18) and (19), we obtain

(20) $$Z_\beta^{1}(w) = O(nw)^{-\alpha-\beta}, \qquad (nw \geqq 1).$$

We are now in a position to consider $Z_\beta^{2}(w)$. We have

$$Z_\beta^{2}(w) = w^{1+q-\alpha}\int_{w}^{\pi}(t-w)^{\alpha-1}t^{-1-q}g_2(n,t)\,dt$$
$$=\int_0^{w}u^{q-\alpha}\int_u^{\pi}(t-u)^{\alpha-1}t^{-q}(d/dt)g_2(n,t)\,dt\,du,$$

since the function $(\beta)_n g_2(n,t) = \sum_{2\nu<n}(\beta-1)_{n-\nu}\sin\nu t$ vanishes at 0 and π. Therefore we may write

(21) $$Z_\beta^{2}(w) = \int_w^{\pi}t^{-q}(dg_2/dt)\int_0^{w}u^{q-\alpha}(t-u)^{\alpha-1}du\,dt$$
$$+\int_0^{w}t^{-q}(dg_2/dt)\int_0^{t}u^{q-\alpha}(t-u)^{\alpha-1}du\,dt.$$

The last term is a constant multiple of

$$(22)\quad \int_0^w (dg_2/dt)\,dt = g_2(n, w) = (1/(\beta)_n) \sum_{2\nu < n} (\beta - 1)_{n-\nu} \sin \nu w$$
$$= (1/(\beta)_n) O(n^{\beta-1}) \max_{\mu} |\sin w + \sin 2w + \cdots + \sin \mu w|$$
$$= O(nw)^{-1},$$

since $(\beta - 1)_{n-\nu}$ is increasing in ν. And it follows from the second mean value theorem that

$$(23)\quad \int_w^\pi (dg_2/dt) \int_0^{w/t} v^{q-\alpha}(1-v)^{\alpha-1} dv\,dt = B(\alpha, 1 + q - \alpha) \int_w^b (dg_2/dt)\,dt$$
$$= O(nw)^{-1}.$$

Combining the results (20), (21), (22) and (23), we obtain

$$(24)\qquad Z_\beta(w) = Z_\beta^1(w) + Z_\beta^2(w) = O(nw)^{-\alpha-\beta}, \qquad (nw \geqq 2).$$

Secondly, let $nw < 2$. Using Lemma 5, we have

$$(25)\quad w^{1+q-\alpha} \int_w^{2w} (t - w)^{\alpha-1} t^{-1-q} g^\beta(n, t)\,dt = w^{1+q-\alpha} \int_w^{2w} (t - w)^{\alpha-1} t^{-1-q} O(nw)\,dt$$
$$= w^{-\alpha} \int_w^{2w} (t - w)^{\alpha-1} O(nw)\,dt = O(nw).$$

By Lemma 5, $g^\beta(n, t) = O(nt)^{-\beta}$ for $t > 0$. It follows that

$$(26)\quad w^{1+q-\alpha} \int_{2w}^\pi (t - w)^{\alpha-1} t^{-1-q} g^\beta(n, t)\,dt = O(n^{-\beta}) w^{1+q-\alpha} \int_{2w}^\pi (t - w)^{\alpha-1} t^{-1-q-\beta}\,dt$$
$$= O(nw)^{-\beta} \int_2^{\pi/w} (v - 1)^{\alpha-1} v^{-1-q-\beta}\,dv = O(nw)^{-\beta},$$

since the integral $\int_2^\infty (v - 1)^{\alpha-1} v^{-1-q-\beta}\,dv$ converges, by the hypothesis $\alpha - \beta < 1 + q$. From (25) and (26) we have

$$(27)\qquad Z_\beta(w) = O(nw)^{-\beta} \qquad (nw \leqq 2).$$

The two relations (24) and (27) establishes the lemma.

12. Proof of Theorem 4. By hypothesis, the functions

$$\psi(t) = t^{-q}[t^q \phi(t)]_{-\alpha} = \frac{1}{\Gamma(1-\alpha)}\, t^{\alpha-q} \frac{d}{dt} \int_0^t \frac{u^q \phi(u)\,du}{(t-u)^\alpha}$$

and $t^{q-\alpha}$ are of bounded variation in $(0, \pi)$. Hence the product

$$t^{q-\alpha}\psi(t)=\frac{d}{dt}\int_0^t\frac{u\phi(u)\,du}{(t-u)^\alpha}=\Gamma(1-\alpha)\,(t^q\phi(t))_{-\alpha}$$

is bounded in $(0,\pi)$. It follows from Lemma 1 that

$$\phi(t)=((t^q\phi(t))_{-\alpha})_\alpha t^{-q}, \tag{28}$$

except for a null set of points t.

Write $\tau_n{}^\beta$ and $\sigma_n{}^\beta$ for the n-th Cesàro mean of order β of

$$(2/\pi)\int_0^\pi\phi(t)\,(d/dt)\sin ntdt \quad\text{and}\quad (2/\pi)\int_0^\pi\phi(t)\cos nt\,dt$$

respectively. An easy calculation gives

$$\sigma_n{}^\beta-\sigma_{n-1}{}^\beta=n^{-1}\tau_n{}^\beta.$$

Assuming $-1<-\alpha<\beta<0$. We have to show that $\Sigma\mid n^{-1}\tau_n{}^\beta\mid$ converges. By (28), we have

$$\tau_n{}^\beta=\int_0^\pi((t^q\phi(t))_{-\alpha})_\alpha t^q\,\frac{dg^\beta(n,t)}{dt}\,dt.$$

As in the proof of Theorem 2, this can be rewritten as

$$\tau_n{}^\beta=\frac{1}{\Gamma(\alpha)}\int_0^\pi w^{\alpha-q}(w^q\phi(w))_{-\alpha}dZ_\beta(w)=-\frac{\sin\alpha\pi}{\pi}\int_0^\pi Z_\beta(w)\,d\psi(w).$$

It follows from Lemma 7 that

$$\mid\tau_n{}^\beta/n\mid\leqq An^{-1}\int_0^{1/n}(nw)^{-\beta}\mid d\psi(w)\mid+An^{-1}\int_{1/n}^\pi(nw)^{-\alpha-\beta}\mid d\psi(w)\mid.$$

On account of $-\beta>0$ and $-\alpha-\beta<0$, the convergence of the series $\Sigma\mid n^{-1}\tau_n{}^\beta\mid$ follows from Lemma 3. This establishes the theorem.

Putting $\alpha=q=p$, it has been shown that the function

$$\frac{d}{dt}\int_0^t\frac{u^p\phi(u)\,du}{(t-u)^p}\qquad(0<p<1)$$

is of bounded variation in $(0,\pi)$, if both $\phi(t)$ and $t\phi'(t)$ are of bounded variation in $(0,\pi)$. It follows from Theorem 4 that the series $\Sigma\mid n^{-1}\tau_n{}^\beta\mid$ converges for $\beta>-p$. Hence the Fourier series is summable $\mid C,\alpha\mid$ for every $\alpha>-1$. We have now obtained the following

THEOREM 5. *If $\phi(t)$ and $t\phi'(t)$ are both of bounded variation in $(0, \pi)$ then the Fourier series of f, at the point x, is absolutely summable (C, α) for every $\alpha > -1$.*

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THE RADICAL AND SEMI-SIMPLICITY FOR ARBITRARY RINGS.* [1]

By N. JACOBSON.

The radical of an algebra with a finite basis, or, more generally, of a ring $\mathfrak{A}$ that satisfies the descending chain condition is defined to be the join of the nil right (left) ideals of $\mathfrak{A}$. The importance of the radical for the structure theory of these rings is due to the facts that 1) the radical $\mathfrak{R}$ is a two-sided ideal whose difference ring $\mathfrak{A} - \mathfrak{R}$ is semi-simple in the sense that its radical is 0, and 2) the structure of semi-simple rings satisfying the descending chain condition can be subjected to a thorough analysis that leads in many important cases to a complete classification. Several investigations of nil ideals in arbitrary rings have been made recently but none of these has led to a structure theory for general semi-simple rings.[2] This is one of a number of indications that in order to develop a satisfactory structure theory for arbitrary rings it is necessary to abandon the concept of a nil ideal in defining the radical.

Other possibilities for defining a radical are afforded by two important characterizations of the radical $\mathfrak{R}$ of an algebra $\mathfrak{A}$ with a finite basis. One of these, due to Perlis, makes use of the notion of quasi-regularity.[3] An element z of $\mathfrak{A}$ is right quasi-regular if there exists a z' in $\mathfrak{A}$ such that $z + z' + zz' = 0$. Perlis has shown that $z \in \mathfrak{R}$ if and only if $u + z$ is right quasi-regular for all right quasi-regular u. A second characterization of $\mathfrak{R}$ for algebras with an identity is that $\mathfrak{R}$ is the intersection of the maximal right (left) ideals of $\mathfrak{A}$.[4] A start in the investigation of the first characterization as a possibility for defining a radical for an arbitrary ring $\mathfrak{A}$ was made by Baer, who showed that the totality $\mathfrak{R}$ of elements z that generate right ideals containing only right quasi-regular elements is a right ideal.[5]

The point of departure of the present investigation is the observation that $\mathfrak{R}$ is a two-sided ideal and that $\mathfrak{R}$ coincides with the two-sided ideal

* Received September 6, 1944; Revised January 5, 1945.

[1] Presented to the Society October 28, 1944.

[2] Baer [1] and Levitzki [1] and [2].

[3] Perlis [1].

[4] Jacobson [1], p. 66. Cf. also G. Birkhoff [1].

[5] Baer [1], p. 562. This definition of the radical has been independently proposed by Hille and Zorn who proved that $\mathfrak{R}$ is a two-sided ideal and that if $\mathfrak{A}$ has an identity, $\mathfrak{R}$ is the intersection of the maximal right (left) ideals of $\mathfrak{A}$. These results were announced by Professor Hille in his Colloquium Lectures in August 1944.

defined in a similar manner using left quasi-regularity and left ideals. We define the radical of $\mathfrak{A}$ to be the ideal $\mathfrak{R}$. In the first part of the paper we establish a number of "radical-like" properties of $\mathfrak{R}$ and we investigate $\mathfrak{R}$ in several important special cases. For rings that satisfy the descending chain condition $\mathfrak{R} = \mathfrak{N}$ where $\mathfrak{N}$ is defined as above. If $\mathfrak{A}$ is a commutative normed ring $\mathfrak{R}$ coincides with the radical as defined by Gelfand to be the totality of generalized nilpotent elements.[6] A similar characterization holds for non-commutative normed rings.

In the latter half of the paper we investigate the relation between $\mathfrak{R}$ and the intersection of the maximal right ideals of $\mathfrak{A}$. We show that if $\mathfrak{A} \neq \mathfrak{R}$ then $\mathfrak{R} \geq \Pi\mathfrak{I}$ for the maximal right ideals $\mathfrak{I}$ of $\mathfrak{A}$ and $\Pi\mathfrak{I} \geq \mathfrak{A}\mathfrak{R}$. If $\mathfrak{A}$ has an identity, $\mathfrak{R} = \Pi\mathfrak{I}$. For arbitrary rings that contain maximal right ideals we show that $\mathfrak{R}$ is the intersection of certain two-sided ideals $\mathfrak{B}$ whose difference rings are of a special type called primitive.

This result implies that any semi-simple ring is isomorphic to a subring of the complete direct sum of primitive rings. To a certain extent this reduces the study of semi-simple rings to that of primitive rings. A tool for studying the latter is a representation theorem for these rings that states that a primitive ring is isomorphic to a dense ring of linear transformations in a vector space over a division ring. These theorems are analogues of the fundamental Wedderburn-Artin structure theorems for semi-simple rings that satisfy the descending chain condition. The Wedderburn-Artin theorems can be deduced quite simply from our results. Moreover, our results contain as special cases Stone's theorem on Boolean rings and other known results on the representability of rings as subrings of direct sums of fields.[7] We can also obtain from our theory a theory of algebraic algebras that is fairly conclusive for algebras with elements of bounded degree. These results will be published in a subsequent paper.

I am indebted to R. Baer for a number of important suggestions that led to simplifications of some of the proofs and to extensions of several of the theorems from rings with an identity to arbitrary rings.

1. Definition of the radical. Let $\mathfrak{A}$ be a ring with an identity and let z be an element of $\mathfrak{A}$ such that $1 + z$ has a right inverse u. We write $u = 1 + z'$. Then $(1 + z)(1 + z') = 1$ implies

$$z + z' + zz' = 0. \tag{1}$$

Conversely if z is an element for which there exists an element z' satisfying (1) then $1 + z$ has the right inverse $1 + z'$. This leads to

[7] Stone [1], McCoy and Montgomery [1], McCoy [1] and [2].
[6] Gelfand [1], p. 10.

Definition 1. An element z of an arbitrary ring $\mathfrak{A}$ is *right quasi-regular* if there exists an element z' in $\mathfrak{A}$ such that $z + z' + zz' = 0$. The element z' satisfying this equation is called a *right quasi-inverse* of z.

We have noted that if $\mathfrak{A}$ has an identity then z is right quasi-regular with right inverse z' if, and only if, $1 + z$ has the right inverse $1 + z'$.

If z is any element of $\mathfrak{A}$ the totality $\{x + zx\}$ of elements $x + zx$ where x ranges over $\mathfrak{A}$ is a right ideal. If z is right quasi-regular with right quasi-inverse z' then $-z = z' + zz' \epsilon \{x + zx\}$. Hence $zx \epsilon \{x + zx\}$ and $x \epsilon \{x + zx\}$. Then $\{x + zx\} = \mathfrak{A}$. On the other hand if $\{x + zx\} = \mathfrak{A}$ then $-z = z' + zz'$ for a suitable z' and z is right quasi-regular. Hence we have the alternative

Definition 1'. An element z of a ring $\mathfrak{A}$ is *right quasi-regular* if the totality of elements $\{x + zx\} = \mathfrak{A}$.[8]

A right ideal $\mathfrak{J}$ will be called *quasi-regular* if all the elements of $\mathfrak{J}$ are right quasi-regular. Let $\mathfrak{J}_1$ and $\mathfrak{J}_2$ be quasi-regular right ideals and let $z_i \epsilon \mathfrak{J}_i$, $i = 1, 2$. There exists an element z'_1 such that $z_1 + z'_1 + z_1z'_1 = 0$. Since $z_2 + z_2z'_1 \epsilon \mathfrak{J}_2$ this element has a right quasi-inverse w' such that

$$(z_2 + z_2z'_1) + w' + (z_2 + z_2z'_1)w' = 0.$$

Hence

$$\begin{aligned}(z_1 + z_2) + (z'_1 + w' + z'_1w') + (z_1 + z_2)(z'_1 + w' + z'_1w') \\ = [z_1 + z'_1 + z_1z'_1] + [(z_2 + z_2z'_1) + w' + (z_2 + z_2z')w'] \\ + [(z_1 + z'_1 + z_1z'_1)w'] = 0.\end{aligned}$$

Thus $z'_1 + w' + z'_1w'$ is a right quasi-inverse of $z_1 + z_2$. Hence $\mathfrak{J}_1 + \mathfrak{J}_2$ is quasi-regular.

Now let $\mathfrak{R}$ be the join of all the quasi-regular right ideals of $\mathfrak{A}$. Since the right ideal generated by an element z is the totality of elements $zi + za$ where i is an integer and $a \epsilon \mathfrak{A}$, it is clear that $\mathfrak{R}$ is the totality of elements z such that $zi + za$ is right quasi-regular for all integral i and all a in $\mathfrak{A}$. The above result shows that $\mathfrak{R}$ is a right ideal. We wish to show that $\mathfrak{R}$ is a two-sided ideal. Let $z \epsilon \mathfrak{R}$ and $b \epsilon \mathfrak{A}$. Then $zb \epsilon \mathfrak{R}$ and there exists an element w' such that $zb + w' + (zb)w' = 0$. Then

$$bz + (-bz - bw'z) + bz(-bz - bw'z) = -b(w' + zb + zbw')z = 0$$

and so $-(bz + bw'z)$ is a right quasi-inverse for bz. Similarly if i is an integer and $a \epsilon \mathfrak{A}$, $(bz)i + (bz)a = b(zi + za)$ is right quasi-regular. Hence $bz \epsilon \mathfrak{R}$ and we have proved

[8] This definition of quasi-regularity is due to Baer.

THEOREM 1. *If $\mathfrak{A}$ is an arbitrary ring the join $\mathfrak{R}$ of all the quasi-regular right ideals of $\mathfrak{A}$ is a (right) quasi-regular two-sided ideal.*

Definition 2. The *radical* of a ring is the join of all the quasi-regular right ideals of the ring.

We have seen that $\mathfrak{R}$ is the set of elements z such that $zi + za$ is right quasi-regular for all integers i and all a in $\mathfrak{A}$. If $\mathfrak{A}$ is a ring with an identity, the connection between quasi-regularity and regularity shows that $\mathfrak{R}$ is the totality of elements z such that $1 + za$ has a right inverse for every a in $\mathfrak{A}$.

A similar discussion holds for left ideals. We define *left quasi-regularity, left quasi-inverse, quasi-regular left ideal* and left radical $\mathfrak{R}'$ in a manner analogous to the above. As for ordinary inverses we say that an element is *quasi-regular* if it is both right and left quasi-regular.

LEMMA 1. *If z is quasi-regular any right (left) quasi-inverse is a left (right) quasi-inverse, is uniquely determined and commutes with z.*

Let z' be a right quasi-inverse and z'' a left quasi-inverse. Then

$$z + z' + zz' = 0 \quad \text{and} \quad z + z'' + z''z = 0.$$

Hence

$$\begin{aligned} z'' &= z'' + (z + z' + zz') + z''(z + z' + zz') \\ &= z' + (z + z'' + z''z) + (z + z'' + z''z)z' = z'. \end{aligned}$$

This proves the first two statements. Since

$$z + z' + zz' = 0 = z + z' + z'z, \qquad zz' = z'z.$$

We call z' *the quasi-inverse* of z.

Now let $z \in \mathfrak{R}$. Then z has a right quasi-inverse $z' = -z - zz'$. Since $\mathfrak{R}$ is an ideal $z' \in \mathfrak{R}$. Hence z' has a right quasi-inverse. Since z is a left quasi-inverse of z' it follows by Lemma 1 that z' is quasi-regular and z is its quasi-inverse. Hence z is quasi-regular. Since $\mathfrak{R}$ is a left ideal $\mathfrak{R} \leq$ the left radical $\mathfrak{R}'$. By symmetry $\mathfrak{R}' \leq \mathfrak{R}$. This proves

THEOREM 2. *The radical of a ring $\mathfrak{A}$ is the join of all the quasi-regular left ideals of $\mathfrak{A}$.*

By the Lemma the elements of $\mathfrak{R}$ are quasi-regular.

2. Elementary properties of the radical. If $\mathfrak{A}$ is an arbitrary ring we know that we can imbed $\mathfrak{A}$ in a ring with an identity $\mathfrak{A}^*$ such that $\mathfrak{A}^* = \mathfrak{A} + (1)$, $\mathfrak{A} \wedge (1) = 0$ where (1), the ring generated by 1, is iso-

morphic to the ring of integers.[9] At first we assume only that $\mathfrak{A}^* = \mathfrak{A} + (1)$. Let $\mathfrak{R}(\mathfrak{A}^*)$ be the radical of $\mathfrak{A}^*$ and let $z \in \mathfrak{R}(\mathfrak{A}^*) \wedge \mathfrak{A}$. Then z has a quasi-inverse z' in $\mathfrak{A}^*$. Since $z' = -z - zz'$, $z' \in \mathfrak{A}$ and z is quasi-regular in $\mathfrak{A}$. Hence $(\mathfrak{R}(\mathfrak{A}^*) \wedge \mathfrak{A}) \leq \mathfrak{R}(\mathfrak{A})$. Since $\mathfrak{A}^* = \mathfrak{A} + (1)$ any right ideal of $\mathfrak{A}$ is a right ideal of $\mathfrak{A}^*$. Hence $\mathfrak{R}(\mathfrak{A}) \leq \mathfrak{R}(\mathfrak{A}^*)$. Thus $\mathfrak{R}(\mathfrak{A}) = \mathfrak{R}(\mathfrak{A}^*) \wedge \mathfrak{A}$.

Suppose now that $\mathfrak{A} \wedge (1) = 0$ and that (1) is isomorphic to the ring of integers. Then if $z^* \in \mathfrak{R}(\mathfrak{A}^*)$ the coset $\bar{z}^*$ of z^* in the difference ring $\mathfrak{A}^* - \mathfrak{A}$ is in the radical of this ring. Since the radical of the ring of integers is 0, $\bar{z}^* = 0$ and $z^* \in \mathfrak{A}$. Hence $\mathfrak{A} \geq \mathfrak{R}(\mathfrak{A}^*)$. From the equation $\mathfrak{R}(\mathfrak{A}) = \mathfrak{R}(\mathfrak{A}^*) \wedge \mathfrak{A}$ we obtain $\mathfrak{R}(\mathfrak{A}) = \mathfrak{R}(\mathfrak{A}^*)$.

THEOREM 3. *Let $\mathfrak{A}$ be an arbitrary ring and let $\mathfrak{A}^*$ be a ring with an identity containing $\mathfrak{A}$ such that $\mathfrak{A}^* = \mathfrak{A} + (1)$. Then the radical $\mathfrak{R}(\mathfrak{A}) = \mathfrak{R}(\mathfrak{A}^*) \wedge \mathfrak{A}$. If in addition $\mathfrak{A} \wedge (1) = 0$ and (1) is isomorphic to the ring of integers then $\mathfrak{R}(\mathfrak{A}) = \mathfrak{R}(\mathfrak{A}^*)$.*

We shall call $\mathfrak{A}$ a *radical ring* if $\mathfrak{A} = \mathfrak{R}$. If $\mathfrak{R} = 0$, $\mathfrak{A}$ is *semi-simple.* For many problems the following theorem effects a reduction to the consideration of these two extreme types of rings.

THEOREM 4. *If $\mathfrak{R}$ is the radical of $\mathfrak{A}$, $\overline{\mathfrak{A}} = \mathfrak{A} - \mathfrak{R}$ is semi-simple.*

Let $\bar{z}$ be an element of the radical of $\overline{\mathfrak{A}}$ and let z be an element in the coset $\bar{z}$. Then there exists an element z' such that $z + z' + zz' = u \in \mathfrak{R}$. Also there exists a u' such that $u + u' + uu' = 0$. Hence

$$\begin{aligned} 0 &= (z + z' + zz') + u' + (z + z' + zz')u' \\ &= z + (z' + u' + z'u') + z(z' + u' + z'u'). \end{aligned}$$

Thus z is right quasi-regular. Since the totality of elements z in the cosets $\bar{z}$ of $\mathfrak{R}(\overline{\mathfrak{A}})$ is an ideal, this totality is a quasi-regular ideal. Hence $z \in \mathfrak{R}$ and $\bar{z} = 0$.

If z is a nilpotent element of index n, $z' = \sum_{1}^{n-1} (-1)^i z^i$ is a quasi-inverse of z. Hence we have the following

THEOREM 5. *The radical of a ring contains every nil right (left) ideal of the ring.*

As a consequence of Theorems 4 and 5 we may prove the

COROLLARY. *If z is an element such that $\mathfrak{A}z\mathfrak{A} \leq \mathfrak{R}$ then $z \in \mathfrak{R}$.*

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[9] Albert [1], p. 22.

For if $\mathfrak{J}$ is the right ideal generated by z, $\mathfrak{J}^3 \leq \mathfrak{A}z\mathfrak{A} \leq \mathfrak{R}$. Hence $\bar{\mathfrak{J}} = (\mathfrak{J} + \mathfrak{R}) - \mathfrak{R}$ is nilpotent in the semi-simple ring $\bar{\mathfrak{A}} = \mathfrak{A} - \mathfrak{R}$. Hence $\bar{\mathfrak{J}} = 0$, $\mathfrak{J} \leq \mathfrak{R}$ and $z \,\epsilon\, \mathfrak{R}$.

This corollary implies that $z \,\epsilon\, \mathfrak{R}$ if and only if za (az) is right (left) quasi-regular for all a in $\mathfrak{A}$.

The elements of the radical need not be nilpotent. This can be seen in the example: $\mathfrak{A}$ the ring of p-adic integers. By applying the definition directly or by using Corollary 2 to Theorem 18 we can show that $\mathfrak{R} = p\mathfrak{A}$. However, no element of $\mathfrak{A}$ is nilpotent. Some information on the nature of the elements of $\mathfrak{R}$ can be obtained from the following

THEOREM 6. *If $\mathfrak{N}$ is a subring of $\mathfrak{R}$ and $z \,\epsilon\, \mathfrak{N}$, then for any positive integer h either $z^{h-1}\mathfrak{N} > z^h\mathfrak{N}$ or $z^h = 0$.*

Evidently $z^{h-1}\mathfrak{N} \geq z^h\mathfrak{N}$. Suppose that $z^{h-1}\mathfrak{N} = z^h\mathfrak{N}$. Then $z^h = z^h y$ for some y in $\mathfrak{N}$. Let y' be a quasi-inverse of $-y$. Then

$$0 = z^h - z^h y + z^h y' - z^h y y' = z^h + z^h(-y + y' - yy') = z^h.$$

This theorem implies that the radical contains no idempotent element $\neq 0$. It is known that if $\mathfrak{A}$ is a ring with an identity whose lattice of right ideals is completely reducible, then every right ideal $\mathfrak{J}$ in $\mathfrak{A}$ has the form $e\mathfrak{A}$ where e is an idempotent element in $\mathfrak{J}$.[10] Hence we have the following

THEOREM 7. *If $\mathfrak{A}$ is a ring with an identity whose lattice of right (left) ideals is completely reducible, then $\mathfrak{A}$ is semi-simple.*

We shall show next that any ring which is regular[11] in the sense of von Neumann is semi-simple. We recall the definition: $\mathfrak{A}$ is regular if every element a of $\mathfrak{A}$ has a *relative inverse* u such that $aua = a$. Suppose that $a \,\epsilon\, \mathfrak{R}$. Then $-ua$ has a quasi-inverse v such that $-ua + v - uav = 0$. Hence

$$0 = -aua + av - auav = -a + av - av = -a.$$

THEOREM 8. *Any regular ring is semi-simple.*

3. The radical of a ring satisfying the descending chain condition. We suppose now that $\mathfrak{A}$ is a ring for which the descending chain condition for right (left) ideals holds. Let $\mathfrak{N}$ be a two-sided ideal contained in $\mathfrak{R}$ and suppose that $\mathfrak{N}^2 = \mathfrak{N}$. Then if $\mathfrak{N} \neq 0$ there exists a minimum right ideal $\mathfrak{J}$ of $\mathfrak{A}$ with the properties 1) $\mathfrak{J} \leq \mathfrak{N}$, 2) $\mathfrak{J}\mathfrak{N} \neq 0$.

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[10] See, for example, Jacobson [1], p. 65.

[11] Von Neumann [1].

Let b be an element of $\mathfrak{J}$ such that $b\mathfrak{N} \neq 0$. Then $(b\mathfrak{N})\mathfrak{N} = b\mathfrak{N} \neq 0$ and since $b\mathfrak{N} \leq \mathfrak{J}$ we have $b\mathfrak{N} = \mathfrak{J}$ by the minimality of $\mathfrak{J}$. Since $b \in \mathfrak{J}$ there is an element y in $\mathfrak{N}$ such that $by = b$. As in the proof of Theorem 6 this leads to $b = 0$ contrary to $b\mathfrak{N} \neq 0$. Thus $\mathfrak{N} = 0$. Now the positive integral powers $\mathfrak{R}^k$ of $\mathfrak{R}$ are two-sided ideals and $\mathfrak{R} \geq \mathfrak{R}^2 \geq \cdots$. Hence there is an integer ρ such that $\mathfrak{R}^\rho = \mathfrak{R}^{\rho+1}$. Then for $\mathfrak{N} = \mathfrak{R}^\rho$ we have $\mathfrak{N}^2 = \mathfrak{N}$. Hence $\mathfrak{N} = \mathfrak{R}^\rho = 0$. This proves

THEOREM 9. *If $\mathfrak{A}$ is a ring that satisfies the descending chain condition for right (left) ideals, then the radical of $\mathfrak{A}$ is nilpotent.*

Since any nil ideal is contained in the radical this proves that any nil ideal in a ring that satisfies the descending chain condition for right (left) ideals is nilpotent. It is clear also that $\mathfrak{R}$ coincides with the usual radical defined as the join of all nilpotent ideals.

4. Finitely generated ideals contained in $\mathfrak{R}$.

THEOREM 10. *If $\mathfrak{N}$ is a right ideal with a finite basis contained in the radical $\mathfrak{R}$ then either $\mathfrak{N}\mathfrak{R} < \mathfrak{N}$ or $\mathfrak{N} = 0$.*

Since $\mathfrak{N}$ is a right ideal either $\mathfrak{N}\mathfrak{R} < \mathfrak{N}$ or $\mathfrak{N}\mathfrak{R} = \mathfrak{N}$. Assume that $\mathfrak{N}\mathfrak{R} = \mathfrak{N}$. Let $y_1, \cdots, y_n$ be a basis for $\mathfrak{N}$. Then every element of $\mathfrak{N}$ has the form $\Sigma y_i a_i + \Sigma y_i j_i$ where the $a_i \in \mathfrak{A}$ and the j_i are integers. Since $\mathfrak{N}\mathfrak{R} = \mathfrak{N}$ every element of $\mathfrak{N}$ also has the form $\Sigma y_i z_i$, z_i in $\mathfrak{R}$. In particular $y_1 = \Sigma y_i z_i$. Let z'_1 be the quasi-inverse of $-z_1$. Then we have

$$y_1 = y_1 + y_1(-z_1 + z'_1 - z_1 z'_1)$$
$$= (y_1 - y_1 z_1) + (y_1 - y_1 z_1) z'_1 = \sum_2^n y_j(z_j + z_j z'_1).$$

Hence y_1 can be eliminated from the basis. Similarly every y_i can be eliminated and so $\mathfrak{N} = 0$.

If $\mathfrak{A}$ is an arbitrary ring we define the transfinite powers $\mathfrak{A}^\alpha$ by the conditions 1) $\mathfrak{A}^1 = \mathfrak{A}$, 2) $\mathfrak{A}^{\alpha+1} = \mathfrak{A}^\alpha \mathfrak{A}$, 3) if α is the limit ordinal $\mathfrak{A}^\alpha$ is the join of all $\mathfrak{A}^\beta$ with $\beta < \alpha$. There exists a least ordinal ρ such that $\mathfrak{A}^\rho = \mathfrak{A}^{\rho+1}$. We shall call ρ the *index* of $\mathfrak{A}$ and we shall say that $\mathfrak{A}$ is *transfinite nilpotent* if $\mathfrak{A}^\rho = 0$. Now suppose that $\mathfrak{A}$ is a ring that satisfies the ascending chain condition for right ideals. We recall that this condition is equivalent to the requirement that every ideal has a finite basis. Let $\mathfrak{N} = \mathfrak{R}^\rho$ where ρ is the index of the radical $\mathfrak{R}$ of $\mathfrak{A}$. Then $\mathfrak{N}\mathfrak{R} = \mathfrak{N}$ and so, by Theorem 10, $\mathfrak{N} = \mathfrak{R}^\rho = 0$.

THEOREM 11. *The radical of a ring that satisfies the ascending chain condition for right ideals is transfinite nilpotent.*

5. The radical of a matrix ring. If $\mathfrak{A}$ is an arbitrary ring we denote as usual the ring of $n \times n$ matrices with elements in $\mathfrak{A}$ by $\mathfrak{A}_n$. If $\mathfrak{B}$ is a subring (ideal) in $\mathfrak{A}$ then $\mathfrak{B}_n$ is a subring (ideal) in $\mathfrak{A}_n$. Let $\mathfrak{R}$ be the radical of $\mathfrak{A}$. Then we wish to show that $\mathfrak{R}_n$ is the radical $\mathfrak{R}(\mathfrak{A}_n)$ of $\mathfrak{A}_n$.

LEMMA 2. *Any matrix $z = (z_{ij})$ of $\mathfrak{A}_n$ in which z_{11} is right quasi-regular and the $z_{ij} = 0$ for $i > 1$ is right quasi-regular in $\mathfrak{A}_n$.*

Since z_{11} is right quasi-regular we know that the ideal $\{x + z_{11}x\} = \mathfrak{A}$. Hence there exist elements z'_{1i} such that $z'_{1i} + z_{11}z'_{1i} = -z_{1i}$. Then if we set $z'_{ij} = 0$ for $i > 1$ we may verify directly that $z' = (z'_{ij})$ is a right quasi-inverse of z.

Consider now the totality $\mathfrak{J}_1$ of matrices with first row consisting of elements in $\mathfrak{R}$ and other rows 0. Then $\mathfrak{J}_1$ is a right ideal. Hence by the lemma $\mathfrak{J}_1 \leq \mathfrak{R}(\mathfrak{A}_n)$. Similarly the totality $\mathfrak{J}_j$ of matrices with j-th row consisting of elements of $\mathfrak{R}$ and other rows 0 is contained in $\mathfrak{R}(\mathfrak{A}_n)$. Since

$$\mathfrak{R}_n = \mathfrak{J}_1 + \cdots + \mathfrak{J}_n, \qquad \mathfrak{R}_n \leq \mathfrak{R}(\mathfrak{A}_n).$$

Conversely let $Y = (y_{ij}) \in \mathfrak{R}(\mathfrak{A}_n)$. If a is any element of $\mathfrak{A}$ we denote the matrix that has a in the (i, j) position and 0's elsewhere by A_{ij}. Let a and b be arbitrary and form the matrix $D = \sum_k A_{kp}YB_{qk}$. Then D is the diagonal matrix $\{ay_{pq}b, \cdots, ay_{pq}b\}$ and $D \in \mathfrak{R}(\mathfrak{A}_n)$. If $D' = (d'_{ij})$ is a right quasi-inverse of D it is easy to see that $d' = d'_{11}$ is a right quasi-inverse of $d = ay_{pq}b$. Evidently this implies that for arbitrary c in $\mathfrak{A}$, and arbitrary integral i, $dc + di$ is right quasi-regular. Hence $d \in \mathfrak{R}$. Since a and b are arbitrary the Corollary to Theorem 5 shows that $y_{pq} \in \mathfrak{R}$ and $Y \in \mathfrak{R}_n$.

THEOREM 12. *If $\mathfrak{A}$ is an arbitrary ring and $\mathfrak{A}_n$ is the ring of $n \times n$ matrices with elements in $\mathfrak{A}$, then the radical $\mathfrak{R}(\mathfrak{A}_n) = \mathfrak{R}_n$, $\mathfrak{R}$ the radical of $\mathfrak{A}$.*

6. The radical of an algebra. Let $\mathfrak{A}$ be an algebra of possibly infinite order over a field Φ. By an ideal in the algebra $\mathfrak{A}$ we mean, of course, an ideal of the ring $\mathfrak{A}$ that is invariant under the scalar multiplications $x \to x\alpha$, α in Φ. If $\mathfrak{A}$ has an identity, $x\alpha = x(1\alpha) = (1\alpha)x$ and so any ideal of the ring $\mathfrak{A}$ is an ideal of the algebra $\mathfrak{A}$.

The above discussion is valid without change for an arbitrary algebra $\mathfrak{A}$. Thus the radical $\mathfrak{R}$ of $\mathfrak{A}$ can be defined to be the join of all the quasi-regular right ideals of $\mathfrak{A}$. $\mathfrak{R}$ is also the maximum quasi-regular left ideal of $\mathfrak{A}$. All of our theorems hold for algebras.

An element a of an algebra $\mathfrak{A}$ is *algebraic* if it satisfies a non-trivial algebraic equation with coefficients in the underlying field Φ. An equivalent

condition is that a generates a subalgebra A with a finite basis. As in the special case of a field, an element which is not algebraic will be called *transcendental*. If every element of $\mathfrak{A}$ is algebraic, then $\mathfrak{A}$ is algebraic. If a is an algebraic element and A is the subalgebra generated by a then there exists a positive integer h such that $a^{h-1}A = a^hA$. Hence if a is in the radical, by Theorem 6, a is nilpotent. This proves

THEOREM 13. *If $\mathfrak{A}$ is an algebra over a field, the elements of the radical $\mathfrak{R}$ of $\mathfrak{A}$ are either nilpotent or transcendental over Φ.*

If $\mathfrak{A}$ is algebraic every z in $\mathfrak{R}$ is nilpotent. Hence, since $\mathfrak{R}$ contains every nil ideal, we have the following

THEOREM 14. *The radical of an algebraic algebra $\mathfrak{A}$ is the join of all nil right (left) ideals of $\mathfrak{A}$.*

If $\mathfrak{A}$ is commutative, an element z generates a nil ideal if and only if it is nilpotent. Hence we have the

COROLLARY. *If $\mathfrak{A}$ is a commutative algebraic algebra, the radical of $\mathfrak{A}$ is the totality of its nilpotent elements.*

7. The radical of a normed ring. We suppose now that $\mathfrak{A}$ is a normed ring, i. e., $\mathfrak{A}$ is an algebra over the field Φ of complex numbers and for each a in $\mathfrak{A}$ there is defined a non-negative real norm $\| a \|$ such that

1. $\| a \| > 0$ if $a \neq 0$ $\quad \| 0 \| = 0$
2. $\| a + b \| \leq \| a \| + \| b \|$
3. $\| a\alpha \| = \| a \| \cdot | \alpha |$ if $\alpha \in \Phi$
4. $\| ab \| \leq \| a \| \cdot \| b \|$
5. $\mathfrak{A}$ has an identity and $\| 1 \| = 1$
6. $\mathfrak{A}$ is complete relative to the metric $D(a, b) = \| a - b \|$

Following Gelfand we call an element of $\mathfrak{A}$ a *generalized nilpotent element* if $\lim \| z^n \|^{1/n} = 0$.[12] For commutative normed rings Gelfand has defined the radical to be the totality of generalized nilpotent elements. We shall show that this set coincides with the radical as defined here and we shall obtain a similar characterization of the radical for non-commutative normed rings. We prove first the following

THEOREM 15. *The radical of a normed ring $\mathfrak{A}$ is the totality of elements z such that $(za)^n \to 0$ $((az)^n \to 0)$ for every a in $\mathfrak{A}$.*

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[12] Gelfand [1], p. 10.

Since $\mathfrak{A}$ has an identity $\mathfrak{R}$ is the totality of elements z such that $1 + za$ has an inverse for every a in $\mathfrak{A}$. Now suppose that z is an element such that $(za)^n \to 0$ for every a. Then for any α in Φ $\| (z\alpha)^n \| < 1$ for n sufficiently large. Thus $\| z^n \| < \beta^n$ where $\beta = 1/|\alpha|$. We choose α so that $|\alpha| > 1$. The series $1 - z + z^2 - \cdots$ is ultimately dominated by the convergent series $1 + \beta + \beta^2 + \cdots$. Hence $1 - z + z^2 - \cdots$ exists in $\mathfrak{A}$ and this element is the inverse of $1 + z$. Similarly we can show that if $z' = za$, then $1 + z'$ has an inverse. Hence $z \in \mathfrak{R}$. Conversely suppose that $z \in \mathfrak{R}$. Then $1 + z\alpha$ has an inverse $(1 + z\alpha)^{-1}$ for every α in Φ. Using the fact that $(1 + z\alpha)^{-1}$ is an analytic function of α we may prove, exactly as Gelfand has done in the commutative case, that $(z\alpha)^n \to 0$.[13] In particular $z^n \to 0$ and since $\mathfrak{R}$ is an ideal $(za)^n \to 0$ for every a.

We shall call an ideal a *generalized nil ideal* if all of its elements are generalized nilpotent elements. If $z \in \mathfrak{R}$, $(z\alpha)^n \to 0$. Then for n sufficiently large $\| (z\alpha)^n \| < 1$ and $\| z^n \| < \beta^n$ for $\beta = 1/|\alpha|$. Hence $0 \leq \| z^n \|^{1/n} < \beta$. Since β is arbitrary $\lim \| z^n \|^{1/n} = 0$. Thus z is a generalized nilpotent element and $\mathfrak{R}$ is a generalized nil ideal. Next let $\mathfrak{J}$ be an arbitrary generalized nil right ideal. Then if $y \in \mathfrak{J}$, $\| y^n \|^{1/n} \to 0$. Hence $y^n \to 0$. Since $\mathfrak{J}$ is a right ideal, $y' = ya \in \mathfrak{J}$ and $(y')^n \to 0$. By Theorem 15, $y \in \mathfrak{R}$ and so $\mathfrak{J} \leq \mathfrak{R}$. This proves

THEOREM 16. *The radical of a normed ring is a generalized nil ideal that contains every generalized nil right (left) ideal of the ring.*

Let $\mathfrak{A}$ be commutative. Then if z and $a \in \mathfrak{A}$,

$$\| (za)^n \| = \| z^n a^n \| \leq \| z^n \| \cdot \| a^n \| \leq \| z^n \| \cdot \| a \|^n.$$

Hence $\| (za)^n \|^{1/n} \leq \| z^n \|^{1/n} \cdot \| a \|$ and if z is a generalized nilpotent element then za is a generalized nilpotent element. Thus any generalized nilpotent element generates a generalized nil ideal and $\mathfrak{R}$ is the totality of generalized nilpotent elements.

8. Quotient ideals. We return to the consideration of an arbitrary ring $\mathfrak{A}$. The results that we obtain are also valid for algebras but we shall not state them explicitly for these.

Let $\mathfrak{J}$ be a right ideal in $\mathfrak{A}$. Then if $a \in \mathfrak{A}$ the right multiplication $x \to xa$ determined by a induces an endomorphism $\bar{a}$ in the difference group $\mathfrak{M} = \mathfrak{A} - \mathfrak{J}$. The mapping $\bar{a}$ sends the coset $x + \mathfrak{J}$ into $xa + \mathfrak{J}$. The totality of elements $\bar{a}$ is a subring $\bar{\mathfrak{A}}$ of the ring of endomorphisms of $\mathfrak{M}$ and the

[13] Gelfand [1], p. 10.

correspondence $a \to \bar{a}$ is a homomorphism between $\mathfrak{A}$ and $\bar{\mathfrak{A}}$. The kernel of this homomorphism is a two-sided ideal $\mathfrak{J}:\mathfrak{A}$ which we shall call the *quotient* of $\mathfrak{J}$ relative to $\mathfrak{A}$. Evidently $\mathfrak{A}(\mathfrak{J}:\mathfrak{A}) \leq \mathfrak{J}$ and if $\mathfrak{A}$ has an identity, $(\mathfrak{J}:\mathfrak{A})$ is the largest two-sided ideal of $\mathfrak{A}$ contained in $\mathfrak{J}$. By the fundamental theorem on homomorphisms $\bar{\mathfrak{A}} \cong \mathfrak{A} - (\mathfrak{J}:\mathfrak{A})$.

Let $\mathfrak{J}$ be maximal, i. e., $\mathfrak{A} > \mathfrak{J}$ and there is no right ideal $\mathfrak{J}'$ such that $\mathfrak{A} > \mathfrak{J}' > \mathfrak{J}$. Then $\mathfrak{M} = \mathfrak{A} - \mathfrak{J}$ is irreducible relative to $\bar{\mathfrak{A}}$. As usual we call a ring of endomorphisms $\bar{\mathfrak{A}}$ irreducible if the group $\mathfrak{M}$ in which $\bar{\mathfrak{A}}$ acts is irreducible. Let $\bar{\mathfrak{A}} \neq 0$ have this property. Then the totality $\mathfrak{Z}$ of elements z in $\mathfrak{M}$ such that $z\bar{\mathfrak{A}} = 0$ is a subgroup of $\mathfrak{M}$ invariant under $\mathfrak{A}$. Hence either $\mathfrak{Z} = 0$ or $\mathfrak{Z} = \mathfrak{M}$. Since $\bar{\mathfrak{A}} \neq 0$, $\mathfrak{Z} \neq \mathfrak{M}$ and so $\mathfrak{Z} = 0$. It follows that if x is any element $\neq 0$ of $\mathfrak{M}$ then $x\bar{\mathfrak{A}} \neq 0$. Since $x\bar{\mathfrak{A}}$ is a subgroup invariant under $\bar{\mathfrak{A}}$, $x\bar{\mathfrak{A}} = \mathfrak{M}$. We use this to prove

THEOREM 17. *Any irreducible ring of endomorphisms is semi-simple.*

Let $\bar{z}$ be an element of the radical of $\bar{\mathfrak{A}}$ and let $x \neq 0$ be arbitrary in $\mathfrak{M}$. If $x\bar{z} \neq 0$, $(x\bar{z})\bar{\mathfrak{A}} = \mathfrak{M}$. Hence there is an $\bar{a}$ in $\mathfrak{A}$ such that $x\bar{z}\bar{a} = x$. The element $-\bar{z}\bar{a}$ has a quasi-inverse $\bar{z}'$. Hence

$$x = x - (x\bar{z}\bar{a} - x\bar{z}' + x\bar{z}\bar{a}\bar{z}') = x - x\bar{z}\bar{a} + (x - x\bar{z}\bar{a})\bar{z}' = 0.$$

This contradiction shows that $x\bar{z} = 0$ for all x. Thus $\bar{z} = 0$ and $\bar{\mathfrak{A}}$ is semi-simple.

This theorem has the following important consequence.

COROLLARY. *If $\mathfrak{J}$ is a maximal right ideal $(\mathfrak{J}:\mathfrak{A})$ contains the radical $\mathfrak{R}$ of $\mathfrak{A}$.*

If $(\mathfrak{J}:\mathfrak{A}) = \mathfrak{A}$ there is nothing to prove. Hence suppose that $(\mathfrak{J}:\mathfrak{A}) \neq \mathfrak{A}$. Then $\bar{\mathfrak{A}} \cong \mathfrak{A} - (\mathfrak{J}:\mathfrak{A})$ is an irreducible ring of endomorphisms $\neq 0$. If $z \in \mathfrak{R}$ the coset $\bar{z} = z + (\mathfrak{J}:\mathfrak{A})$ is in the radical of $\mathfrak{A} - (\mathfrak{J}:\mathfrak{A})$. Since $\bar{\mathfrak{A}}$ is semi-simple, $\bar{z} = 0$. Hence $z \in (\mathfrak{J}:\mathfrak{A})$ and $\mathfrak{R} \leq (\mathfrak{J}:\mathfrak{A})$.

9. The radical as intersection of maximal right ideals. Let $\mathfrak{A}$ be any ring that is not a radical ring. Then $\mathfrak{A}$ contains an element a which is not right quasi-regular. Hence the right ideal $\{x + ax\}$ does not contain a. Moreover, if $\mathfrak{J}$ is a right ideal that contains a and contains the ideal $\{x + ax\}$ then $\mathfrak{J} = \mathfrak{A}$. By using Zorn's maximum principle we may prove

LEMMA 3. *If a is an element of $\mathfrak{A}$ that is not right quasi-regular the right ideal $\{x + ax\}$ can be imbedded in a maximal right ideal.*

This shows, of course, that any ring that is not a radical ring contains maximal right ideals. We assume now that $\mathfrak{A}$ is any ring that contains maximal right ideals. Consider the intersection $\Pi\mathfrak{J}$ of the maximal right ideals $\mathfrak{J}$ of $\mathfrak{A}$. Let $y \in \Pi\mathfrak{J}$. Then y is right quasi-regular. For otherwise $\{x + yx\}$ can be imbedded in the maximal right ideal $\mathfrak{J}$. Then $y \in \mathfrak{J}$ and $\mathfrak{J} = \mathfrak{A}$ contrary to the maximality of $\mathfrak{J}$. Thus every element of $\Pi\mathfrak{J}$ is quasi-regular and since $\Pi\mathfrak{J}$ is a right ideal, $\Pi\mathfrak{J} \leqq \mathfrak{R}$. On the other hand, by the corollary to Theorem 17, $\mathfrak{R} \leq (\mathfrak{J}:\mathfrak{A})$. Hence $\mathfrak{A}\mathfrak{R} \leq \mathfrak{A}(\mathfrak{J}:\mathfrak{A}) \leq \mathfrak{J}$ and $\mathfrak{A}\mathfrak{R} \leq \Pi\mathfrak{J}$. This proves

THEOREM 18. *If $\mathfrak{A}$ is a ring that contains maximal right ideals and $\Pi\mathfrak{J}$ is the intersection of these maximal right ideals, then $\Pi\mathfrak{J} \leq \mathfrak{R}$ and $\mathfrak{A}\mathfrak{R} \leq \Pi\mathfrak{J}$.*

COROLLARY 1. *If $\mathfrak{A}$ is not a radical ring and $\Pi\mathfrak{J}$ is the intersection of the maximal right ideals of $\mathfrak{A}$ then $\Pi\mathfrak{J} \leq \mathfrak{R}$ and $\mathfrak{A}\mathfrak{R} \leq \Pi\mathfrak{J}$.*

COROLLARY 2. *If $\mathfrak{A}$ is a ring with an identity the radical of $\mathfrak{A}$ is the intersection of the maximal right ideals of $\mathfrak{A}$.*

For $\mathfrak{A}\mathfrak{R} = \mathfrak{R}$. Hence $\mathfrak{R} \leq \Pi\mathfrak{J}$ as well as $\Pi\mathfrak{J} \leq \mathfrak{R}$.

The following results also are consequences of Theorem 18:

If $\mathfrak{A}$ is a ring that contains maximal right ideals then $\Pi\mathfrak{J}$ is a two-sided ideal.

For $\mathfrak{A}(\Pi\mathfrak{J}) \leq \mathfrak{A}\mathfrak{R} \leq \Pi\mathfrak{J}$. Hence $\Pi\mathfrak{J}$ is a left ideal as well as a right ideal.

The radical of a normed ring is a closed ideal.

It is known that any maximal ideal $\mathfrak{J}$ is closed.[14] Hence $\mathfrak{R} = \Pi\mathfrak{J}$ is closed.

If $\mathfrak{A}$ is a semi-simple ring, $\Pi\mathfrak{J} = 0$. Suppose, in addition, that $\mathfrak{A}$ satisfies the descending chain condition for right ideals. Then we can find a finite number of maximal right ideals $\mathfrak{J}_j$, $j = 1, \cdots, n$ such that $\Pi\mathfrak{J}_j = 0$. We may suppose that the set $\{\mathfrak{J}_j\}$ is minimal in the sense that $\prod_{k \neq i} \mathfrak{J}_k = \mathfrak{J}'_i \neq 0$ for every i. Then $\mathfrak{J}_i \wedge \mathfrak{J}'_i = 0$ and $\mathfrak{J}'_i \nleqq \mathfrak{J}_i$. Since $\mathfrak{J}_i$ is maximal it follows that $\mathfrak{A} = \mathfrak{J}_i + \mathfrak{J}'_i$. Hence $\mathfrak{J}'_i$ is isomorphic to the difference $\mathfrak{A}$-group $\mathfrak{A} - \mathfrak{J}_i$ and $\mathfrak{J}'_i$ is minimal. Using a simple lattice-theoretic argument we can conclude that $\mathfrak{A} = \mathfrak{J}'_1 \oplus \cdots \oplus \mathfrak{J}'_n$.[15] This proves the well-known

THEOREM 19. *If $\mathfrak{A}$ is a semi-simple ring that satisfies the descending chain condition for right ideals, $\mathfrak{A}$ is a direct sum of a finite number of minimal right ideals.*

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[14] Gelfand [1], p. 8. [15] Jacobson [1], p. 35.

Suppose again that $\mathfrak{A}$ is any ring that contains maximal right ideals. Then, by the corollary to Theorem 17, $\mathfrak{R} \leqq \Pi(\mathfrak{J}:\mathfrak{A})$ for all maximal $\mathfrak{J}$. Conversely let $y \in \Pi(\mathfrak{J}:\mathfrak{A})$. Then $\mathfrak{A}y \leq \Pi\mathfrak{J} \leq \mathfrak{R}$. Hence $\mathfrak{A}y\mathfrak{A} \leq \mathfrak{R}$ and this implies that $y \in \mathfrak{R}$. We have therefore proved

THEOREM 20. *Let $\mathfrak{A}$ be an arbitrary ring that contains maximal right ideals. Then the radical of $\mathfrak{A}$ is the intersection $\Pi(\mathfrak{J}:\mathfrak{A})$ where $\mathfrak{J}$ ranges over the maximal right ideals of $\mathfrak{A}$.*

COROLLARY. *If $\mathfrak{A}$ is not a radical ring, $\mathfrak{R} = \Pi(\mathfrak{J}:\mathfrak{A})$ where $\mathfrak{J}$ ranges over the maximal right ideals.*

The results of this section hold also for left ideals. An interesting consequence of the second corollary to Theorem 18 is that if $\mathfrak{A}$ is a ring with an identity then the intersection of the maximal right ideals of $\mathfrak{A}$ coincides with the intersection of the maximal left ideals of $\mathfrak{A}$.

10. Primitive rings.

Definition 3. A ring $\mathfrak{A}$ is *primitive* if $\mathfrak{A}$ contains a maximal right ideal $\mathfrak{J}$ whose quotient $(\mathfrak{J}:\mathfrak{A}) = 0$.

Let $\mathfrak{A}$ be of this type and as before let $\bar{\mathfrak{A}}$ denote the ring of endomorphisms $x + \mathfrak{J} \rightarrow xa + \mathfrak{J}$ in $\mathfrak{M} = \mathfrak{A} - \mathfrak{J}$. Then $\bar{\mathfrak{A}}$ is irreducible and $\bar{\mathfrak{A}} \cong \mathfrak{A} - (\mathfrak{J}:\mathfrak{A}) = \mathfrak{A}$. Thus any primitive ring is isomorphic to an irreducible ring of endomorphisms.

Conversely suppose that $\bar{\mathfrak{A}} \neq 0$ is an irreducible ring of endomorphisms acting in $\mathfrak{M}$. Let x be an element $\neq 0$ in $\mathfrak{M}$ and let $\mathfrak{J}_x$ denote the totality of elements b of $\bar{\mathfrak{A}}$ such that $xb = 0$. Evidently $\mathfrak{J}_x$ is a right ideal in $\bar{\mathfrak{A}}$. We have seen that $x\bar{\mathfrak{A}} = \mathfrak{M}$. Hence $\mathfrak{J}_x < \bar{\mathfrak{A}}$. Let $\bar{a}$ be an element of $\bar{\mathfrak{A}}$ not in $\mathfrak{J}_x$ and let $\bar{c}$ be arbitrary. Then $x\bar{a} \neq 0$. Hence $(x\bar{a})\bar{\mathfrak{A}} = \mathfrak{M}$ and so there exists an element $\bar{u}$ such that $x\bar{a}\bar{u} = x\bar{c}$. Thus $\bar{c} - \bar{a}\bar{u} \in \mathfrak{J}_x$ and $\bar{\mathfrak{A}} = \mathfrak{J}_x + \bar{a}\bar{\mathfrak{A}}$ for any $\bar{a}$ not in $\mathfrak{J}_x$. This proves that $\mathfrak{J}_x$ is maximal. If $\bar{c} \neq 0 \in \bar{\mathfrak{A}}$ there is a y in $\mathfrak{M}$ such that $y\bar{c} \neq 0$. There exists an $\bar{a}$ such that $x\bar{a} = y$. It follows that $x\bar{a}\bar{c} \neq 0$. Thus for any $\bar{c} \neq 0$ there exists an $\bar{a}$ such that $\bar{a}\bar{c} \notin \mathfrak{J}_x$. On the other hand, if $\bar{c} \in (\mathfrak{J}_x:\bar{\mathfrak{A}})$, $\bar{a}\bar{c} \in \mathfrak{J}_x$ for all $\bar{a}$. Hence $\bar{c} = 0$ and $(\mathfrak{J}_x:\bar{\mathfrak{A}}) = 0$.

THEOREM 21. *A necessary and sufficient condition that $\mathfrak{A}$ be primitive is that $\mathfrak{A}$ be isomorphic to an irreducible ring of endomorphisms.*

If $\bar{\mathfrak{B}}$ is a two-sided ideal $\neq 0$ in the irreducible ring $\bar{\mathfrak{A}}$, the argument that we have used before for $\bar{\mathfrak{A}}$ shows that $x\bar{\mathfrak{B}} = \mathfrak{M}$ for any $x \neq 0$. Thus $\bar{\mathfrak{B}}$ is irreducible. Hence we have the following

THEOREM 22. *Any two-sided ideal in a primitive ring is primitive.*

For a deeper study of primitive rings we require a theorem on irreducible rings of endomorphisms recently proved by the author. We recall first that if $\bar{\mathfrak{A}}$ is irreducible in $\mathfrak{M}$ then the totality of endomorphisms of $\mathfrak{M}$ that commute with all the $\bar{a} \in \bar{\mathfrak{A}}$ is a division subring $\mathfrak{D}$ of the ring of endomorphisms of $\mathfrak{M}$.[16] Since $\mathfrak{D}$ contains the identity mapping, we may regard $\mathfrak{M}$ as a vector space over $\mathfrak{D}$.

We call a set $\bar{\mathfrak{A}}$ of linear transformations in a vector space $\mathfrak{M}$ over $\mathfrak{D}$ a *dense set* if for any finite set $x_1, \cdots, x_n$ of $\mathfrak{D}$-independent vectors and any finite set $y_1, \cdots, y_n$ of arbitrary vectors there exists a linear transformation $\bar{a} \in \bar{\mathfrak{A}}$ such that $x_i \bar{a} = y_i$. An equivalent condition is that if $\mathfrak{S}$ is any finite dimensional subspace of $\mathfrak{M}$ and A is any linear transformation in $\mathfrak{S}$ over $\mathfrak{D}$, then A can be extended to a linear transformation $\bar{a} \in \bar{\mathfrak{A}}$ of $\mathfrak{M}$ over $\mathfrak{D}$.[17] Hence if $\mathfrak{M}$ is finite dimensional $\bar{\mathfrak{A}} = \mathfrak{L}$ the complete ring of linear transformations in $\mathfrak{M}$ over $\mathfrak{D}$. We have proved elsewhere the following

THEOREM 23. *If $\bar{\mathfrak{A}} \neq 0$ is an irreducible ring of endomorphisms in the commutative group $\mathfrak{M}$ and $\mathfrak{D}$ is the division ring of endomorphisms commutative with the $\bar{a} \in \bar{\mathfrak{A}}$, then $\bar{\mathfrak{A}}$ is a dense ring of linear transformations in the vector space $\mathfrak{M}$ over $\mathfrak{D}$.*

The converse is trivial. Thus the concepts of primitive ring, irreducible ring of endomorphisms and dense ring of linear transformations may be used interchangeably.

If $\bar{\mathfrak{A}}$ is a dense ring of linear transformations in $\mathfrak{M}$ over $\mathfrak{D}$ and $x_1, x_2, x_3, \cdots$ are $\mathfrak{D}$-independent, then

$$\mathfrak{J}_{x_1} > (\mathfrak{J}_{x_1} \wedge \mathfrak{J}_{x_2}) > (\mathfrak{J}_{x_1} \wedge \mathfrak{J}_{x_2} \wedge \mathfrak{J}_{x_3}) > \cdots$$

where $\mathfrak{J}_x$ denotes the right ideal of annihilators of x. Hence if $\mathfrak{M}$ is infinite dimensional, then $\bar{\mathfrak{A}}$ does not satisfy the descending chain condition for right ideals. Thus if $\bar{\mathfrak{A}}$ satisfies the descending chain condition $\mathfrak{M}$ is finite dimensional over $\mathfrak{D}$ and therefore $\bar{\mathfrak{A}} = \mathfrak{L}$ the complete ring of linear transformations in $\mathfrak{M}$ over $\mathfrak{D}$. This proves

THEOREM 24. *Any primitive ring that satisfies the descending chain condition for right ideals is isomorphic to the complete ring of linear transformations of a suitable finite dimensional vector space over a division ring.*

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[16] This is Schur's lemma. See Jacobson [1], p. 57.

[17] For the definitions and results on dense rings of linear transformations quoted here, see Jacobson [2].

Since the complete ring of linear transformations in a finite dimensional vector space is simple we have the

COROLLARY. *Any primitive ring that satisfies the descending chain condition for right ideals is simple.*

11. Structure of semi-simple rings. We have seen that if $\mathfrak{A}$ is not a radical ring, the radical $\mathfrak{R} = \Pi(\mathfrak{J}:\mathfrak{A})$ for the maximal right ideals $\mathfrak{J}$. The difference rings $\mathfrak{A} - (\mathfrak{J}:\mathfrak{A})$ are isomorphic to irreducible rings of endomorphisms. Hence they are primitive. In particular we have the following

THEOREM 25. *If $\mathfrak{A}$ is a semi-simple ring $\mathfrak{A}$ contains two-sided ideals $\mathfrak{B}$ such that* 1) *each $\mathfrak{A} - \mathfrak{B}$ is primitive and* 2) $\Pi\mathfrak{B} = 0$.

The converse holds also. For if $\mathfrak{B}$ is a two-sided ideal in a ring and $\mathfrak{A} - \mathfrak{B}$ is semi-simple then $\mathfrak{B} \geq \mathfrak{R}$. Since any primitive ring is semi-simple each ideal $\mathfrak{B}$ such that $\mathfrak{A} - \mathfrak{B}$ is primitive contains $\mathfrak{R}$. Hence if $\Pi\mathfrak{B} = 0$ $\mathfrak{R} = 0$.

Theorem 25 reduces, to a certain extent, the study of semi-simple rings to that of primitive rings and hence to that of dense rings of linear transformations in a vector space over a division ring. This will be clearer in the next section where we derive a certain alternate form of this theorem. Before doing this we obtain some consequences of the theorem in the present form. We note first

THEOREM 26. *Any two-sided ideal in a semi-simple ring is semi-simple.*

Let $\mathfrak{A}_1$ be a two-sided ideal in the semi-simple ring $\mathfrak{A}$ and let $\{\mathfrak{B}\}$ be a set of two-sided ideals in $\mathfrak{A}$ such that 1) each $\mathfrak{A} - \mathfrak{B}$ is primitive and 2) $\Pi\mathfrak{B} = 0$. Let $\mathfrak{B}_1 = \mathfrak{A}_1 \wedge \mathfrak{B}$. Then $\mathfrak{A}_1 - (\mathfrak{A}_1 \wedge \mathfrak{B}) \cong (\mathfrak{A}_1 + \mathfrak{B}) - \mathfrak{B}$ a two-sided ideal in the primitive ring $\mathfrak{A} - \mathfrak{B}$. It follows that $\mathfrak{A}_1 - \mathfrak{B}_1$ is primitive. Evidently $\Pi\mathfrak{B} = 0$. Hence $\mathfrak{A}_1$ is semi-simple.

Next let $\mathfrak{A}$ be a semi-simple ring that satisfies the descending chain condition for right ideals. Let $\{\mathfrak{B}\}$ be a set of two-sided ideals satisfying the conditions of Theorem 25. We may replace the set $\{\mathfrak{B}\}$ by a finite set $\mathfrak{B}_i$, $i = 1, \cdots, s$, and we may suppose that the set is minimal in the sense that $\mathfrak{B}'_i = \prod_{j \neq i} \mathfrak{B}_j \neq 0$. Since $\mathfrak{A} - \mathfrak{B}$ is primitive and satisfies the descending chain condition for right ideals, $\mathfrak{A} - \mathfrak{B}$ is simple and so $\mathfrak{B}$ is a maximal two-sided ideal in $\mathfrak{A}$. It follows that $\mathfrak{B}'_i + \mathfrak{B}_i = \mathfrak{A}$. Hence $\mathfrak{B}'_i \cong \mathfrak{A} - \mathfrak{B}_i$ is simple. As for one-sided ideals we can prove that $\mathfrak{A} = \mathfrak{B}'_1 \oplus \cdots \oplus \mathfrak{B}'_s$. This proves the Wedderburn-Artin structure theorem:

THEOREM 27. *Any semi-simple ring that satisfies the descending chain condition for right ideals is isomorphic to a direct sum of a finite number of simple rings.*

By the preceding section the structure of the component ring $\mathfrak{B}'_i$ is that of a ring of linear transformations in a finite dimensional vector space over a division ring. This is the second fundamental Wedderburn-Artin structure theorem.

12. Infinite direct sums. Let S be an arbitrary set. We call the elements of S points and we suppose that to each point P there is associated a ring $\mathfrak{E}_P$. Now let $\mathfrak{S} = (S, \mathfrak{E}_P)$ be the totality of functions $f(P)$ with domain S and with $f(P)$ in $\mathfrak{E}_P$. If $f(P)$ and $g(P) \in \mathfrak{S}$ we define $f + g$ by $(f + g)(P) = f(P) + g(P)$ and we define $fg(P) = f(P)g(P)$. Then it is readily verified that $\mathfrak{S}$ is a ring. We shall call it the *complete direct sum* of the rings $\mathfrak{E}_P$. Let $\mathfrak{S}_0$ be the subset of functions $f(P)$ such that $f(P) = 0$ for all but a finite number of the points P. Then $\mathfrak{S}_0$ is a subring of $\mathfrak{S}$. We call it the (discrete) *direct sum* of the rings $\mathfrak{E}_P$. Let $\overline{\mathfrak{E}}_Q$ be the subset of $\mathfrak{S}_0$ of elements $f(P)$ such that $f(P) = 0$ for all $P \neq Q$. Then 1) $\overline{\mathfrak{E}}_Q$ is a two-sided ideal in $\mathfrak{S}_0$, 2) $\mathfrak{S}_0 = \Sigma\overline{\mathfrak{E}}_Q$ where $\Sigma\overline{\mathfrak{E}}_Q$ denotes the totality of finite sums of the elements of the ideals $\overline{\mathfrak{E}}_Q$ and 3) $\overline{\mathfrak{E}}_Q \wedge \sum_{P\neq Q} \overline{\mathfrak{E}}_P = 0$. Conversely these conditions are sufficient that a ring be isomorphic to a direct sum of rings isomorphic to the ideals $\overline{\mathfrak{E}}_Q$.

If $\mathfrak{A}$ is a subring of the complete direct sum $\mathfrak{S}$ we define the *component* $\mathfrak{A}_P$ *of* $\mathfrak{A}$ *at* P to be the totality of elements $f(P)$ of $\mathfrak{E}_P$ for f in $\mathfrak{A}$. It is clear that $\mathfrak{A}_P$ is a subring of $\mathfrak{E}_P$ and the correspondence $f \to f(P)$ is a homomorphism between $\mathfrak{A}$ and $\mathfrak{A}_P$. The kernel of this homomorphism is the two-sided ideal $\mathfrak{B}_P$ of elements g such that $g(P) = 0$. Hence $\Pi\mathfrak{B}_P = 0$. Moreover, $\mathfrak{A}_P \cong \mathfrak{A} - \mathfrak{B}_P$.

We suppose, conversely, that S is a set of two-sided ideals $P = \mathfrak{B}$ such that $\Pi\mathfrak{B} = 0$. Let $\mathfrak{A}_P = \mathfrak{A} - \mathfrak{B}$ and form the complete direct sum $\mathfrak{S} = (S, \mathfrak{A}_P)$. Then each a in $\mathfrak{A}$ determines an element $\bar{a}$ in $\mathfrak{S}$ such that $\bar{a}(P)$ is the coset $a + \mathfrak{B}$. The set $\overline{\mathfrak{A}}$ of the $\bar{a}$ is a subring of $\mathfrak{S}$ and the correspondence $a \to \bar{a}$ is a homomorphism. It is clear that component $\overline{\mathfrak{A}}_P = \mathfrak{A}_P$ and that $\bar{a}(P) = 0$ if and only if $a \in \mathfrak{B}$. Thus $\bar{a} = 0$ if and only if $a \in \Pi\mathfrak{B}$ and since $\Pi\mathfrak{B} = 0$ we see that the correspondence $a \to \bar{a}$ is, in fact, an isomorphism between $\mathfrak{A}$ and $\overline{\mathfrak{A}}$. Using these remarks we may formulate Theorem 25 and its converse as follows:

THEOREM 28. *A necessary and sufficient condition that a ring* $\mathfrak{A}$ *be semi-*

simple is that $\mathfrak{A}$ *be isomorphic to a subring* $\bar{\mathfrak{A}}$ *of a complete direct sum* $(S, \bar{\mathfrak{A}}_P)$ *where the components* $\bar{\mathfrak{A}}_P$ *of* $\bar{\mathfrak{A}}$ *are primitive.*

This theorem serves as a substitute in the general theory for the Wedderburn-Artin Theorem 27. A number of special cases of the theorem are well-known. We cite, for example, Stone's theorem on the representability of a Boolean ring as a subring of a direct sum of the fields $\mathfrak{J}_2$ of residues mod 2 and Montgomery and McCoy's generalization of this theorem to commutative rings in which every element a satisfies the equations $pa = 0$, $a^p = a$ for a fixed prime p. These results follow from Theorem 28 and the easily proved fact that any primitive commutative ring is a field.

13. Primitive rings that contain minimal ideals. If $\mathfrak{A}$ is an arbitrary ring it is well-known that any minimal right ideal $\mathfrak{J}$ in $\mathfrak{A}$ either has the form $e\mathfrak{A}$ where e is an idempotent element or $\mathfrak{J}^2 = 0$.[18] If $\mathfrak{J}$ is a minimal right ideal and a is arbitrary $a\mathfrak{J}$ is either 0 or a minimal right ideal. It follows that the join $\Sigma\mathfrak{J}$ of all the minimal right ideals is a two-sided ideal.[19]

If $\bar{\mathfrak{A}}$ is an irreducible ring of endomorphisms in the commutative group $\mathfrak{M}$ and $\bar{\mathfrak{B}}$ is a two-sided ideal $\neq 0$ in $\bar{\mathfrak{A}}$ we have seen that $x\bar{\mathfrak{B}} = \mathfrak{M}$ for every $x \neq 0$ in $\mathfrak{M}$. $\bar{\mathfrak{B}}$ is primitive also. If $\bar{\mathfrak{B}}_1$ and $\bar{\mathfrak{B}}_2$ are two-sided ideals $\neq 0$ then the equation $x\bar{\mathfrak{B}}_1 = \mathfrak{M}$ implies that $\bar{\mathfrak{B}}_1\bar{\mathfrak{B}}_2 \neq 0$. We state this as

LEMMA 4. *If* $\mathfrak{B}_1, \mathfrak{B}_2$ *are two-sided ideals* $\neq 0$ *in the primitive ring* $\mathfrak{A}$, $\mathfrak{B}_1\mathfrak{B}_2 \neq 0$.

This, of course, implies that $\mathfrak{B}_1 \wedge \mathfrak{B}_2 \neq 0$.

Suppose now that $\mathfrak{A}$ is a primitive ring that contains minimal right ideals. Since $\mathfrak{A}$ is semi-simple any minimal right ideal $\mathfrak{J}$ of $\mathfrak{A}$ has the form $e\mathfrak{A}$, $e^2 = e \neq 0$. Let $\mathfrak{F}$ be the two-sided ideal $\Sigma\mathfrak{J}$ for all minimal right ideals $\mathfrak{J}$. We assert that $\mathfrak{F}$ is a minimal two-sided ideal. For let $\mathfrak{F}_1$ be a two-sided ideal such that $\mathfrak{F}_1 < \mathfrak{F}$. There exists a minimal right ideal $\mathfrak{J} \nleq \mathfrak{F}_1$. Since $\mathfrak{J} \wedge \mathfrak{F}_1$ is a right ideal and $\mathfrak{J}$ is minimal it follows that $\mathfrak{J} \wedge \mathfrak{F}_1 = 0$. Hence also $\mathfrak{J}\mathfrak{F}_1 = 0$. Since $\mathfrak{J}$ is not nilpotent $\mathfrak{A}\mathfrak{J} \neq 0$. Evidently $\mathfrak{A}\mathfrak{J}$ is a two-sided ideal and since $(\mathfrak{A}\mathfrak{J})\mathfrak{F}_1 = 0$, $\mathfrak{F}_1 = 0$ by the lemma. The minimality of $\mathfrak{F}$ has therefore been established. It follows from Lemma 4 that $\mathfrak{F}$ is contained in every two-sided ideal $\mathfrak{B} \neq 0$ of $\mathfrak{A}$. We wish to show next that $\mathfrak{F}$ is a simple ring. Let $\mathfrak{F}_1 < \mathfrak{F}$ now denote a two-sided ideal of $\mathfrak{F}$. Then $\mathfrak{F}\mathfrak{F}_1\mathfrak{F} \leq \mathfrak{F}_1 < \mathfrak{F}$ and $\mathfrak{F}\mathfrak{F}_1\mathfrak{F}$ is a two-sided ideal of $\mathfrak{A}$. Hence $\mathfrak{F}\mathfrak{F}_1\mathfrak{F} = 0$ and since $\mathfrak{F}$ is primitive $\mathfrak{F}_1 = 0$.

[18] See, for example, Jacobson [1], p. 64.

[19] This ideal has been called the anti-radical by Baer. See Baer [1], p. 544.

THEOREM 29. *Let $\mathfrak{A}$ be a primitive ring that contains minimal right ideals and let $\mathfrak{F} = \Sigma\mathfrak{J}$ the join of all the minimal right ideals of $\mathfrak{A}$. Then $\mathfrak{F}$ is a two-sided ideal that is contained in every two-sided ideal $\neq 0$ of $\mathfrak{A}$ and $\mathfrak{F}$ is a simple ring.*

If $\mathfrak{B}$ is a two-sided ideal $\neq 0$ of $\mathfrak{A}$, $\mathfrak{B}$ contains every minimal right ideal $\mathfrak{J}$ of $\mathfrak{A}$. It is easy to see by the above reasoning that $\mathfrak{J}$ is minimal in $\mathfrak{B}$. Hence $\mathfrak{B}$ satisfies the same conditions as $\mathfrak{A}$.

Again let $\bar{\mathfrak{A}}$ be an irreducible ring of endomorphisms in a commutative group $\mathfrak{M}$ and let $\mathfrak{D}$ be the division ring of endomorphisms in $\mathfrak{M}$ commutative with the elements of $\bar{\mathfrak{A}}$. We call a linear transformation A in a vector space $\mathfrak{M}$ over $\mathfrak{D}$ *finite valued* if the image space $\mathfrak{M}A$ is finite dimensional. We have shown elsewhere that if $\bar{\mathfrak{A}}$ contains finite valued linear transformations $\neq 0$ then $\bar{\mathfrak{A}}$ contains minimal right ideals.[20] Conversely suppose that $\bar{\mathfrak{A}}$ contains a minimal right ideal $\mathfrak{J}$. Then $\mathfrak{J} = \bar{e}\bar{\mathfrak{A}}$; $\bar{e}^2 = \bar{e} \neq 0$. We wish to show that $\mathfrak{M}\bar{e}$ is one dimensional. For, if not, there exist two $\mathfrak{D}$-independent elements x, y of $\mathfrak{M}$ such that $x\bar{e} = x$, $y\bar{e} = y$. By the density of $\bar{\mathfrak{A}}$ there exists a $\bar{b}$ in $\bar{\mathfrak{A}}$ such that $x\bar{b} = 0$, $y\bar{b} \neq 0$. Then $\bar{c} = \bar{e}\bar{b}$ satisfies $x\bar{c} = 0$, $y\bar{c} \neq 0$ and $\bar{c} \in \mathfrak{J}$. Hence if $\mathfrak{J}_x$ denotes the annihilator of x, $\mathfrak{J}_x \wedge \mathfrak{J} \neq 0$. Since $\mathfrak{J}$ is minimal $\mathfrak{J} = \mathfrak{J} \wedge \mathfrak{J}_x$. But this contradicts the fact that $\bar{e} \in \mathfrak{J}$ and $x\bar{e} \neq 0$.

We have shown also that if $\bar{\mathfrak{A}}$ contains finite valued transformations $\neq 0$ the totality $\bar{\mathfrak{F}}$ of these transformations is a two-sided ideal contained in every two-sided ideal $\neq 0$ of $\mathfrak{A}$.[21] Thus $\bar{\mathfrak{F}} = \Sigma\mathfrak{J}$ for all minimal right ideals $\mathfrak{J}$. This completes the proof of the following

THEOREM 30. *Let $\bar{\mathfrak{A}} \neq 0$ be an irreducible ring of endomorphisms in $\mathfrak{M}$ and let $\mathfrak{D}$ be the division ring of endomorphisms commutative with the elements of $\bar{\mathfrak{A}}$. Then $\bar{\mathfrak{A}}$ contains minimal right ideals if and only if it contains finite valued linear transformations $\neq 0$ of $\mathfrak{M}$ over $\mathfrak{D}$. If the condition is satisfied the minimal two-sided ideal $\bar{\mathfrak{F}}$ coincides with the totality of finite valued transformations in $\bar{\mathfrak{A}}$.*

We recall that a homomorphism $a \rightarrow \bar{a}$ between an abstract ring $\mathfrak{A}$ and a ring of endomorphisms $\bar{\mathfrak{A}}$ in a commutative group $\mathfrak{M}$ is called a *representation* of $\mathfrak{A}$. The representation is *irreducible* if $\bar{\mathfrak{A}}$ is irreducible. Two representations are *equivalent* if the groups in which the endomorphisms act are $\mathfrak{A}$-isomorphic. We have seen that the fundamental fact about a primitive ring is that it possesses a $(1-1)$ irreducible representation.

[20] Jacobson [2], p. 232. [21] Jacobson [2], p. 230.

Suppose now that $\mathfrak{A}$ is a primitive ring that contains a minimal right ideal $\mathfrak{J}$ and let $a \rightarrow \bar{a}$ be any $(1-1)$ irreducible representation of $\mathfrak{A}$. Let $\bar{\mathfrak{A}}$ be the corresponding ring of endomorphisms and $\mathfrak{M}$ the group in which they act. The image $\bar{\mathfrak{J}}$ of $\mathfrak{J}$ is $\neq 0$. Hence there exists an x in $\mathfrak{M}$ such that $x\bar{\mathfrak{J}} \neq 0$. Because of the irreducibility of $\mathfrak{M}$, $x\bar{\mathfrak{J}} = \mathfrak{M}$. It can be verified directly that the correspondence $u \rightarrow xu$, u in $\mathfrak{J}$, is an $\mathfrak{A}$-isomorphism between $\mathfrak{J}$ and $\mathfrak{M}$. Thus the representation in $\mathfrak{M}$ is equivalent to the representation by the right multiplications in $\mathfrak{J}$. This proves

THEOREM 31. *If $\mathfrak{A}$ is a primitive ring that contains minimal right ideals, any two $(1-1)$ irreducible representations of $\mathfrak{A}$ are equivalent.*

Now let $\bar{\mathfrak{A}}_1$ and $\bar{\mathfrak{A}}_2$ be dense rings of linear transformations in $\mathfrak{M}_1$ over $\mathfrak{D}_1$ and $\mathfrak{M}_2$ over $\mathfrak{D}_2$ respectively. Suppose that the $\bar{\mathfrak{A}}_i$ are isomorphic under a correspondence $\bar{a}_1 \rightarrow \bar{a}_2$ and that the $\bar{\mathfrak{A}}_i$ contain minimal right ideals. Then if $\mathfrak{A}$ is the abstract primitive ring isomorphic to the $\bar{\mathfrak{A}}_i$ we have two irreducible representations $a \rightarrow \bar{a}_1$, $a \rightarrow \bar{a}_2$ of $\mathfrak{A}$. Since these representations are equivalent we have a $(1-1)$ correspondence $S : x_1 \rightarrow x_2 = x_1 S$ between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ such that

$$(x_1 + y_1)S = x_1 S + y_1 S, \qquad (x_1 \bar{a}_1)S = (x_1 S)\bar{a}_2.$$

The latter equation may be written in the form $\bar{a}_2 = S^{-1}\bar{a}_1 S$. Now it is easy to see that $\mathfrak{D}_i$ is the complete set of endomorphisms in $\mathfrak{M}_i$ commutative with $\mathfrak{A}_i$.[22] Hence the correspondence $\alpha_1 \rightarrow S^{-1}\alpha_1 S = \alpha_2$ for α_1 in $\mathfrak{D}_1$ is an isomorphism between $\mathfrak{D}_1$ and $\mathfrak{D}_2$. This shows that the two vector spaces are isomorphic. If we identify these spaces, setting $\mathfrak{M}_i = \mathfrak{M}$, $\mathfrak{D}_i = \mathfrak{D}$, then S becomes a semi-linear transformation in $\mathfrak{M}$ over $\mathfrak{D}$.

THEOREM 32. *Two dense rings of linear transformations that contain minimal right ideals can not be isomorphic unless the spaces in which they act are isomorphic. If $\bar{\mathfrak{A}}_1$ and $\bar{\mathfrak{A}}_2$ are two dense rings in $\mathfrak{M}$ over $\mathfrak{D}$ and the $\bar{\mathfrak{A}}_i$ contain minimal right ideals and are isomorphic under a correspondence $\bar{a}_1 \rightarrow \bar{a}_2$ then there exists a $(1-1)$ semi-linear transformation S in $\mathfrak{M}$ over $\mathfrak{D}$ such that $\bar{a}_2 = S^{-1}\bar{a}_1 S$ for all $\bar{a}_1$.*

This result shows that for primitive rings that contain minimal right ideals we have precisely the same uniqueness of representation as a dense ring of linear transformations as we have in the classical case of simple rings that satisfy the descending chain condition.

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[22] Jacobson [2], p. 233.

14. Atomic semi-simple rings. Let $\mathfrak{A}$ be any ring that contains minimal right ideals and let $\mathfrak{F}$ be the join $\Sigma\mathfrak{J}$ of all the minimal right ideals $\mathfrak{J}$ of $\mathfrak{A}$. The right multiplications $u \to ua$ in $\mathfrak{J}$ form an irreducible ring and therefore a semi-simple ring. It follows that if $\mathfrak{R}$ is the radical of $\mathfrak{A}$ then $\mathfrak{J}\mathfrak{R} = 0$. Hence $\mathfrak{F}\mathfrak{R} = 0$.

We shall call a ring $\mathfrak{A}$ *atomic* if $\mathfrak{A} = \mathfrak{F}$. If $\mathfrak{A}$ is atomic $\mathfrak{A}$ is semi-simple if and only if $\mathfrak{A}$ contains no nilpotent ideals. Let $\mathfrak{A}$ be of this type and let $\mathfrak{J}$ be a minimal right ideal in $\mathfrak{A}$. Then we have the following

LEMMA 5. *If $\mathfrak{A}$ is atomic and semi-simple and $\mathfrak{J}$ is a minimal right ideal in $\mathfrak{A}$ then $\mathfrak{A}\mathfrak{J} \geq \mathfrak{J}$ and $\mathfrak{A}\mathfrak{J}$ is a simple ring.*

If $\mathfrak{B}$ is any two-sided ideal of $\mathfrak{A}$ either $\mathfrak{B} \geq \mathfrak{J}$ or $\mathfrak{J} \wedge \mathfrak{B} = 0 = \mathfrak{J}\mathfrak{B}$. Now if $\mathfrak{J}(\mathfrak{A}\mathfrak{J}) = 0$, $\mathfrak{J}^3 = 0$ contrary to the semi-simplicity. Hence $\mathfrak{A}\mathfrak{J} \geq \mathfrak{J}$. If $\mathfrak{B}$ is any two-sided ideal of $\mathfrak{A}$ properly contained in $\mathfrak{A}\mathfrak{J}$, $\mathfrak{B} \not\geq \mathfrak{J}$. Hence $\mathfrak{J}\mathfrak{B} = 0$, $(\mathfrak{A}\mathfrak{J})\mathfrak{B} = 0$ and $\mathfrak{B}^2 = 0$. Hence $\mathfrak{B} = 0$. Thus $\mathfrak{A}\mathfrak{J}$ is a minimal two-sided ideal of $\mathfrak{A}$. It follows readily that $\mathfrak{A}\mathfrak{J}$ is a simple ring.

Let $\{\mathfrak{A}_\alpha\}$ be the set of distinct two-sided ideals $\mathfrak{A}\mathfrak{J}$, $\mathfrak{J}$ minimal. Each $\mathfrak{A}_\alpha$ is simple. Hence if $\alpha \neq \beta$, $\mathfrak{A}_\alpha\mathfrak{A}_\beta = 0$. It follows that $\mathfrak{A}_\alpha \wedge \sum_{\beta \neq \alpha} \mathfrak{A}_\beta = 0$. By the lemma $\mathfrak{A} = \Sigma\mathfrak{A}_\alpha$. Hence we have proved

THEOREM 33. *If $\mathfrak{A}$ is atomic and semi-simple $\mathfrak{A}$ is isomorphic to a direct sum of simple rings.*

Each component $\mathfrak{A}\mathfrak{J}$ is a join of minimal right ideals $a\mathfrak{J}$ of $\mathfrak{J}$. It is easy to see that these ideals are also minimal in $\mathfrak{A}\mathfrak{J}$. Hence by Theorem 29 each $\mathfrak{A}\mathfrak{J}$ is isomorphic to a dense ring of finite-valued linear transformations.

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ON EVEN-DIMENSIONAL SKEW-METRIC SPACES AND THEIR GROUPS OF TRANSFORMATIONS.*

By Hwa-Chung Lee.

In a previous paper [1] we have studied the differential geometry of an even-dimensional space L_{2n} with a non-singular skew-symmetric fundamental tensor $a_{\alpha\beta}$ $(\alpha, \beta, \gamma, \rho, \sigma, \tau = 1, \cdots, 2n)$. We shall here discuss certain properties of the following transformation groups in such a space.

In the group G of *analytic point-transformations* of L_{2n} there is a subgroup G_1 of *conformal transformations* each leaving the fundamental tensor invariant save for a non-vanishing factor. Referring L_{2n} to an arbitrary chosen coordinate system a conformal point transformation $x \to y$ satisfies, by definition, the equation

$$\psi a_{\alpha\beta}(x) = a_{\rho\sigma}(y) \frac{\partial y^\rho}{\partial x^\alpha} \frac{\partial y^\sigma}{\partial x^\beta} \qquad (\psi \neq 0). \tag{1}$$

If the factor ψ is restricted to be constant, we have the smaller subgroup G_2 of *special conformal transformations*. If we require $\psi = 1$, we obtain the further subgroup G_3 of *automorphisms*. Thus

$$G_3 \subset G_2 \subset G_1 \subset G.$$

1. Let us first dispose of the case of an L_2. In this case (1) reduces to the single equation

$$\psi a_{12}(x) = a_{12}(y) \left(\frac{\partial y^1}{\partial x^1} \frac{\partial y^2}{\partial x^2} - \frac{\partial y^2}{\partial x^1} \frac{\partial y^1}{\partial x^2} \right).$$

Since every L_2 is flat [2] we may refer it to a coordinate system in which each component of the fundamental tensor is constant,[3] thus $a_{12}(x) = a_{12}(y) =$ const. $(\neq 0)$, so that

$$\psi = \frac{\partial(y^1, y^2)}{\partial(x^1, x^2)} . \tag{2}$$

Hence

Theorem 1. *For an L_2, the group G_3 of automorphisms is isomorphic*

* Received July 24, 1944.

[1] H. C. Lee, "A kind of even-dimensional differential geometry and its application to exterior calculus," *American Journal of Mathematics*, vol. 65 (1943), p. 433.

[2] *Loc. cit.*, p. 434.

[3] *Loc. cit.*, p. 434.

to the unimodular group, and the special conformal group G_2 is isomorphic to the group of transformations of constant functional determinants. The conformal group G_1 is identical with the group G of all transformations, the factor ψ being entirely unrestricted.

We shall supose $2n > 2$. If we differentiate (1) with respect to x^γ, permute α, β, γ cyclically and add, we find

$$(3)\quad \psi K_{\alpha\beta\gamma}(x) + \frac{\partial\psi}{\partial x^\alpha} a_{\beta\gamma}(x) + \frac{\partial\psi}{\partial x^\beta} a_{\gamma\alpha}(x) + \frac{\partial\psi}{\partial x^\gamma} a_{\alpha\beta}(x) = K_{\rho\sigma\tau}(y) \frac{\partial y^\rho}{\partial x^\alpha} \frac{\partial y^\sigma}{\partial x^\beta} \frac{\partial y}{\partial x}$$

where $K_{\alpha\beta\gamma}$ denotes the curvature tensor.[4] Transvecting (3) by the inverse $a^{\beta\gamma}$ of the fundamental tensor, the result is reducible, in consequence of (1), to the form

$$(4)\qquad K_\alpha(x) - (2n-2)\frac{\partial \log \psi}{\partial x^\alpha} = K_\rho(y)\frac{\partial y^\rho}{\partial x^\alpha}$$

where K_α denotes the curvature vector.[5] If we differentiate (4) with respect to x^β, interchange α, β and subtract, we get

$$(5)\qquad b_{\alpha\beta}(x) = b_{\rho\sigma}(y)\frac{\partial y^\rho}{\partial x^\alpha}\frac{\partial y^\sigma}{\partial x^\beta}$$

where $b_{\alpha\beta}$ denotes the first conformal curvature tensor.[6] (5) is a necessary and sufficient condition for (4) to be integrable when regarded as a differential equation in ψ.

If L_{2n} is conformally flat,[7] the first conformal curvature tensor vanishes[8] so that (5) is satisfied identically. Thus (4) is integrable in ψ. In fact in this case a multiplier ϕ exists (determined save for a constant factor) such that $\phi a_{\alpha\beta}$ is the fundamental tensor of the corresponding flat space, and K_α (being a gradient) is expressed in terms of this ϕ by[9]

$$K_\alpha(x) = (2n-2)\frac{\partial \log \phi(x)}{\partial x^\alpha},$$

whence (4) yields

$$(6)\qquad \psi = \text{const.}\ \frac{\phi(x)}{\phi(y)}.$$

Hence

THEOREM 2. *In a conformally flat space L_{2n} $(2n > 2)$ with fundamental*

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[4] *Loc. cit.*, p. 433.

[5] *Loc. cit.*, p. 434.

[6] *Loc. cit.*, p. 435.

[7] *Loc. cit.*, p. 436.

[8] *Loc. cit.*, p. 436.

[9] *Loc. cit.*, p. 436.

tensor $a_{\alpha\beta}$, the factor ψ of a conformal point-transformation $x \to y$ is determined save for a multiplicative constant:

$$\psi = C \frac{\phi(x)}{\phi(y)},$$

where ϕ is the multiplier (determined to within a constant factor) which makes $\phi a_{\alpha\beta}$ the fundamental tensor of the corresponding flat space.

In this conformally flat case, $a_{\alpha\beta}$ is expressible in terms of ϕ and of a quantity a_α (determined save for an additive gradient) in the form [10]

$$\phi(x) a_{\alpha\beta}(x) = \frac{\partial}{\partial x^\alpha} a_\beta(x) - \frac{\partial}{\partial x^\beta} a_\alpha(x).$$

We can then reduce (1) to

$$\frac{\partial}{\partial x^\alpha} \left\{ C a_\beta(x) - a_\sigma(y) \frac{\partial y^\sigma}{\partial x^\beta} \right\} - \frac{\partial}{\partial x^\beta} \left\{ C a_\alpha(x) - a_\rho(y) \frac{\partial y^\rho}{\partial x^\alpha} \right\} = 0,$$

whence

$$C a_\alpha(x) - a_\rho(y) \frac{\partial y^\rho}{\partial x^\alpha} = \text{gradient}.$$

The gradient on the right may be dropped on account of the additive gradient in a_α. Hence

THEOREM 3. *The conformal group G_1 of a conformally flat space L_{2n} $(2n > 2)$ consists of those point-transformations under which a_α behaves like a covariant pseudovector (vector except for a multiplicative constant):*

$$C a_\alpha(x) = a_\rho(y) \frac{\partial y^\rho}{\partial x^\alpha},$$

where a_α is the quantity (determined to within an additive gradient) in terms of which the fundamental tensor of the corresponding flat space is expressed in the form

$$\frac{\partial}{\partial x^\alpha} a_\beta(x) - \frac{\partial}{\partial x^\beta} a_\alpha(x).$$

If L_{2n} itself is flat, then $\phi(x) = \phi(y) = \text{const.}$ so that (6) implies $\psi = \text{const.}$ Conversely $\psi = \text{const.}$ implies also $\phi = \text{const.}$ and so L_{2n} is flat. Thus

THEOREM 4. *For a flat space L_{2n} $(2n > 2)$ the special conformal group*

[10] *Loc. cit.*, p. 436.

G_2 is identical with the conformal group G_1. Conversely, if for a conformally flat space L_{2n} $(2n > 2)$ G_2 is identical with G_1, the space itself is flat.

Before identifying the group G_3 of automorphisms of a flat L_{2n} we first introduce the generalized Poisson parentheses.

2. Consider in a general L_{2n} two functions of position and form the expression

$$(F_1, F_2) = a^{\alpha\beta} \frac{\partial F_1}{\partial x^\alpha} \frac{\partial F_2}{\partial x^\beta}. \tag{7}$$

Evidently we have

$$(F_1, F_2) = -(F_2, F_1),$$

and it is also easy to verify the identity

$$(F_1, (F_2, F_3)) + (F_2, (F_3, F_1)) + (F_3, (F_1, F_2)) = K^{\alpha\beta\gamma} \frac{\partial F_1}{\partial x^\alpha} \frac{\partial F_2}{\partial x^\beta} \frac{\partial F_3}{\partial x^\gamma}$$

where $K^{\alpha\beta\gamma}$ denotes the contravariant associate of the curvature tensor:

$$K^{\alpha\beta\gamma} = a^{\alpha\rho} a^{\beta\sigma} a^{\gamma\tau} K_{\rho\sigma\tau} = a^{\alpha\rho} \frac{\partial a^{\beta\gamma}}{\partial x^\rho} + a^{\beta\sigma} \frac{\partial a^{\gamma\alpha}}{\partial x^\sigma} + a^{\gamma\tau} \frac{\partial a^{\alpha\beta}}{\partial x^\tau}.$$

The above identity reduces to the Jacobi relation

$$(F_1, (F_2, F_3)) + (F_2, (F_3, F_1)) + (F_3, (F_1, F_2)) = 0$$

when L_{2n} is flat, referred to any coordinate system whatever.

For a flat L_{2n}, there exist coordinate systems in which the components of the fundamental tensor are constants.[11] Such coordinate systems may be called *preferred*. In particular a preferred coordinate system will be called *canonical* when

$$a_{\alpha\beta} = \begin{pmatrix} O & I \\ -I & O \end{pmatrix}, \qquad a^{\alpha\beta} = \begin{pmatrix} O & -I \\ I & O \end{pmatrix}, \tag{8}$$

where I and O denote the unit and zero matrices of order n respectively.

In a flat L_{2n} referred to a canonical coordinate system, (7), on account of (8), may be written in the form

$$(F_1, F_2) = \sum_{i=1}^{n} \left\{ \frac{\partial F_1}{\partial p_i} \frac{\partial F_2}{\partial x^i} - \frac{\partial F_1}{\partial x^i} \frac{\partial F_2}{\partial p_i} \right\} \tag{9}$$

where $p_i = x^{n+i}$ $(i = 1, \cdots, n)$. This is the well known *Poisson parenthesis.*

To consider the group G_3 of automorphisms of a flat L_{2n} put $\psi = 1$ in (1) and write it in the equivalent form

[11] *Loc. cit.*, p. 434.

$$a^{\rho\sigma}(y) = a^{\alpha\beta}(x) \frac{\partial y^\rho}{\partial x^\alpha} \frac{\partial y^\sigma}{\partial x^\beta}. \tag{10}$$

The right-hand side, in canonical coordinates, is no other than the Poisson parenthesis (y^ρ, y^σ). Thus (10) in these coordinates decomposes into

$$(y^i, y^j) = 0, \quad (q_i, y^j) = \delta_i{}^j, \quad (q_i, q_j) = 0, \qquad (i, j = 1, \cdots, n) \tag{11}$$

where $q_i = y^{n+i}$, which are precisely *Poisson's conditions* for a *canonical transformation* [12] or in particular *for a homogeneous contact transformation.*[13] Hence

THEOREM 5. *The group G_3 of automorphisms of a flat L_{2n} is isomorphic to the group of canonical transformations in $2n$ variables.*

We remark that (10) is equivalent to

$$a_{\alpha\beta}(x) = a_{\rho\sigma}(y) \frac{\partial y^\rho}{\partial x^\alpha} \frac{\partial y^\sigma}{\partial x^\beta}$$

(which is just (1) with $\psi = 1$) whose right-hand side is, by (8), of the form

$$\sum_{i=1}^{n} \left\{ \frac{\partial y^i}{\partial x^\alpha} \frac{\partial q_i}{\partial x^\beta} - \frac{\partial q_i}{\partial x^\alpha} \frac{\partial y^i}{\partial x^\beta} \right\}$$

which is *Lagrange's parenthesis.* Thus the above equation in a canonical coordinate system represents *Lagrange's condition* for a canonical transformation. Similarly, (10) is also equivalent to each of the following two

$$a^{\alpha\beta}(x) = a^{\rho\sigma}(y) \frac{\partial x^\alpha}{\partial y^\rho} \frac{\partial x^\beta}{\partial y^\sigma}, \qquad a_{\rho\sigma}(y) = a_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial y^\rho} \frac{\partial x^\beta}{\partial y^\sigma},$$

which in a canonical coordinate system represent the Poisson and Lagrange conditions respectively for the inverse transformation to be a canonical transformation.

3. The special conformal group G_2 of any L_{2n} has a very remarkable property in the following connection. Consider a system of curves in L_{2n} defined by a system of ordinary differential equations of the form

$$\frac{dx^\alpha}{dt} + a^{\alpha\beta}(x) \frac{\partial H(x, t)}{\partial x^\beta} = 0 \tag{12}$$

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[12] See, for example, E. Goursat, *Leçons sur l'intégration des équations aux dérivées partielles du premier ordre,* 2nd ed. (1921), p. 387.

[13] See, for example, L. P. Eisenhart, *Continuous groups of transformations* (1933), p. 250.

where t is a parameter and H is a given function of t and of the x's. We call such a system of curves a *Hamiltonian congruence,* and the function H the *Hamiltonian* of the congruence. These denominations are justified when L_{2n} is flat and is referred to a canonical coordinate system, for in this case (12) may be written on account of (8) in the form

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}, \qquad (i = 1, \cdots, n)$$

which are the well known Hamilton's canonical equations in Analytical Mechanics. By a point-transformation in L_{2n}, a Hamiltonian congruence of curves is transformed into a system of curves which is in general no longer a Hamiltonian congruence. We call a point transformation a *Hamiltonian transformation* if it transforms *every* Hamiltonian congruence into a Hamiltonian congruence. The Hamiltonian transformations in L_{2n} evidently form a group, which we shall now show to be no other than the special conformal group G_2. In other words

THEOREM 6. *The special conformal group G_2 of any space L_{2n} consists of the Hamiltonian transformations of the space.*

To prove this theorem consider a Hamiltonian transformation $x \to y$ which, by definition, transforms (12) for an arbitrary H into an equation of the same form

$$\frac{dy^\rho}{dt} + a^{\rho\sigma}(y) \frac{\partial \bar{H}(y, t)}{\partial y^\sigma} = 0. \tag{13}$$

Since

$$\frac{dy^\rho}{dt} = \frac{\partial y^\rho}{\partial x^\alpha} \frac{dx^\alpha}{dt} = -\frac{\partial y^\rho}{\partial x^\alpha} a^{\alpha\beta}(x) \frac{\partial H(x, t)}{\partial x^\beta}$$

by (12), (13) may be written

$$\frac{\partial \bar{H}}{\partial y^\sigma} = a_{\sigma\rho}(y) \frac{\partial y^\rho}{\partial x^\alpha} a^{\alpha\beta}(x) \frac{\partial H}{\partial x^\beta}. \tag{14}$$

To obtain the conditions of integrability of (14) for $\bar{H}$ we first find by differentiation

$$\frac{\partial}{\partial y^\tau} \left\{ \frac{\partial \bar{H}}{\partial y^\sigma} \right\} = \frac{\partial}{\partial y^\tau} \left\{ a_{\sigma\rho}(y) \frac{\partial y^\rho}{\partial x^\alpha} a^{\alpha\beta}(x) \right\} \frac{\partial H}{\partial x^\beta} + a_{\sigma\rho}(y) \frac{\partial y^\rho}{\partial x^\alpha} a^{\alpha\beta}(x) \frac{\partial x^\gamma}{\partial y^\tau} \frac{\partial}{\partial x^\gamma} \left\{ \right.$$

Subtracting from this equation the equation obtained from it by interchanging

σ and τ, and in the result putting the coefficients of $\partial H/\partial x^\beta$ and $\partial^2 H/\partial x^\gamma \partial x^\beta$ (which latter is symmetric in β, γ) equal to zero (since H is arbitrary), we obtain the required conditions

$$\text{(15)} \qquad \frac{\partial}{\partial y^\tau}\left\{ a_{\sigma\rho}(y)\, \frac{\partial y^\rho}{\partial x^\alpha}\, a^{\alpha\beta}(x) \right\} - \frac{\partial}{\partial y^\sigma}\left\{ a_{\tau\rho}(y)\, \frac{\partial y^\rho}{\partial x^\alpha}\, a^{\alpha\beta}(x) \right\} = 0,$$

$$\text{(16)} \qquad a_{\sigma\rho}(y)\frac{\partial y^\rho}{\partial x^\alpha}\left\{ a^{\alpha\beta}(x)\,\frac{\partial x^\gamma}{\partial y^\tau} + a^{\alpha\gamma}(x)\,\frac{\partial x^\beta}{\partial y^\tau} \right\}$$
$$- a_{\tau\rho}(y)\, \frac{\partial y^\rho}{\partial x^\alpha}\left\{ a^{\alpha\beta}(x)\,\frac{\partial x^\gamma}{\partial y^\sigma} + a^{\alpha\gamma}(x)\,\frac{\partial x^\beta}{\partial y^\sigma} \right\} = 0.$$

To simplify these conditions we first transvect (16) by $\dfrac{\partial y^\sigma}{\partial x^\lambda}\,\dfrac{\partial y^\tau}{\partial x^\mu}$ $(\lambda, \mu = 1, \cdots, 2n)$, finding that (16) is equivalent to

$$\text{(17)} \qquad w_{\lambda\alpha}\{a^{\alpha\beta}(x)\delta_\mu{}^\gamma + a^{\alpha\gamma}(x)\delta_\mu{}^\beta\} - w_{\mu\alpha}\{a^{\alpha\beta}(x)\delta_\lambda{}^\gamma + a^{\alpha\gamma}(x)\delta_\lambda{}^\beta\} = 0$$

where for the moment we put

$$\text{(18)} \qquad w_{\alpha\beta} = a_{\rho\sigma}(y)\, \frac{\partial y^\rho}{\partial x^\alpha}\, \frac{\partial y^\sigma}{\partial x^\beta}\,.$$

Contracting (17) for γ and μ we find

$$w_{\lambda\alpha}a^{\alpha\beta}(x) = \psi\delta_\lambda{}^\beta \qquad (\psi = - w_{\alpha\beta}a^{\alpha\beta}(x)/2n)$$

which may be written

$$\text{(19)} \qquad w_{\alpha\beta} = \psi a_{\alpha\beta}(x).$$

Since this satisfies (17), it is equivalent to (16). Replacing $w_{\alpha\beta}$ in (19) by its definition (18), (19) assumes the form

$$\text{(20)} \qquad \psi a_{\alpha\beta}(x) = a_{\rho\sigma}(y)\, \frac{\partial y^\rho}{\partial x^\alpha}\, \frac{\partial y^\sigma}{\partial x^\beta}\,.$$

Because of (20) we may write (15) thus

$$\frac{\partial}{\partial y^\tau}\left\{ \psi\, \frac{\partial x^\beta}{\partial y^\sigma} \right\} - \frac{\partial}{\partial y^\sigma}\left\{ \psi\, \frac{\partial x^\beta}{\partial y^\tau} \right\} = 0$$

which implies

$$\text{(21)} \qquad \psi = \text{const.}$$

We see from (20) and (21) that the Hamiltonian transformation $x \to y$ in question is a special conformal transformation.

Consider, conversely, a special conformal transformation $x \to y$. It satisfies (20) and (21) by definition. Since the latter are equivalent to (15) and (16), they are, therefore, the conditions of integrability of (14) for $\bar{H}$.

Hence (14) furnishes an $\tilde{H}$ corresponding to an arbitrarily given H. By means of (14), (12) is transformed into (13), whence the special conformal transformation $x \rightarrow y$ is a Hamiltonian transformation. Theorem 6 is proved.

In consequence of (20) we may write (14) in the form

$$\frac{\partial \tilde{H}}{\partial x^a} = \psi \frac{\partial H}{\partial x^a}$$

whence

$$\tilde{H}(y, t) = \psi H(x, t) + \chi(t). \tag{22}$$

The additive term $\chi(t)$, which is an arbitrary function of t, may be omitted, since a Hamiltonian may be increased by any function of t without affecting the corresponding Hamiltonian congruence. Hence

THEOREM 7. *When a Hamiltonian congruence is transformed by a special conformal transformation of factor ψ, the Hamiltonian of the congruence is simply multiplied by this factor ψ.*

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